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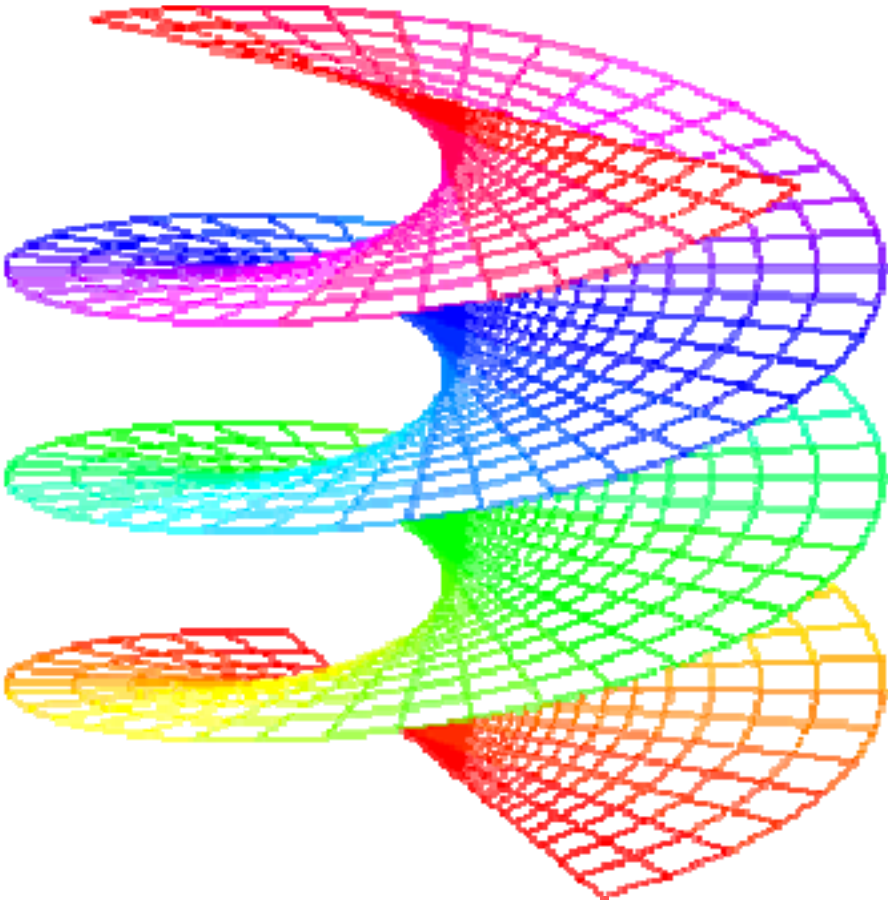

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On the irrational root sieve sequence

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Abstract Let a be any fixed square free number, the irrational root sieve sequence is defined as taking off all k-powers ($k \geq 2$), of all square free numbers a from the natural numbers except 0 and 1. The main purpose of this paper is using the elementary method to study the properties of the irrational root sieve sequence, and give an interesting asymptotic formula for it.

Keywords Irrational root sieve, square free numbers, asymptotic formula.

§1. Introduction and Results

For any positive integer $n > 1$, we call n be a square free number if for any prime p with $p \mid n$, then $p^2 \nmid n$. Now we define the irrational root sieve sequence as follows:

Definition. From the set of all natural numbers (except 0 and 1),

-take off all powers of 2^k, $k \geq 2$ (i.e. 4, 8, 16, 32, 64, $\cdots$);

-take off all powers of 3^k, $k \geq 2$;

-take off all powers of 5^k, $k \geq 2$;

-take off all powers of 6^k, $k \geq 2$;

-take off all powers of 7^k, $k \geq 2$;

-take off all powers of 10^k, $k \geq 2$;

$\cdots$ and so on (take off all k-powers ($k \geq 2$), of all square free numbers).

Now, we can get all irrational root sieve sequence:

2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24, 26, 28, 29, 30, 31, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 65, 66, 67, 68, 69, 71, 72, 73, $\cdots$.

In reference [1], Professor F.Smarandache asked us to study the properties of the irrational root sieve sequence. About this problem, I do not know whether someone had studied it. At least I have not seen any related papers before. In this paper, we use the elementary method to study the this problem, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. Let A denotes the set of all elements of the irrational root sieve sequence. Then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{\substack{a \leq x \\ a \in A}} 1 = x - \frac{6}{\pi^2}\sqrt{x} - \frac{6}{\pi^2}x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}} \cdot \ln x\right).$$

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. First we need the following simple lemma.

Lemma. For any real number $t \geq 1$, we have the asymptotic formula

$$\sum_{n\leq t} |\mu(n)| = \frac{6}{\pi^2}t + O\left(\sqrt{t}\right),$$

where $\mu(n)$ denotes the Möbius function.

Proof. For any real number $t > 1$ and positive integer n, from the properties of the Möbius function $\mu(n)$ (See reference [3]):

$$|\mu(n)| = \sum_{d^2|n} \mu(d)$$

and note that

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},$$

we have

$$\begin{aligned}
\sum_{n\leq t} |\mu(n)| &= \sum_{n\leq t}\sum_{d^2|n} \mu(d) = \sum_{md^2\leq t} \mu(d) \\
&= \sum_{d\leq\sqrt{t}} \mu(d) \sum_{m\leq \frac{t}{d^2}} 1 \\
&= \sum_{d\leq\sqrt{t}} \mu(d) \left(\frac{t}{d^2} + O(1)\right) \\
&= t\left(\sum_{d=1}^{\infty} \frac{\mu(d)}{d^2} - \sum_{d>\sqrt{t}} \frac{\mu(d)}{d^2}\right) + O\left(\sum_{d\leq\sqrt{t}} |\mu(d)|\right) \\
&= t\left(\frac{6}{\pi^2} + O\left(\frac{1}{\sqrt{t}}\right)\right) + O\left(\sqrt{t}\right) \\
&= \frac{6}{\pi^2}t + O\left(\sqrt{t}\right).
\end{aligned}$$

This proves Lemma.

Now we use this Lemma to complete the proof of our theorem. From the definition of A and the above Lemma we have

$$\begin{aligned}
\sum_{\substack{a\leq x\\ a\in A}} 1 &= x - \sum_{\substack{a^k\leq x\\ |\mu(a)|\neq 0\\ k\geq 2\\ a\geq 2}} 1 = x - \sum_{2\leq k\leq \frac{\ln x}{\ln 2}} \sum_{2\leq a\leq \sqrt[k]{x}} |\mu(a)| \\
&= x - \sum_{2\leq a\leq\sqrt{x}} |\mu(a)| - \sum_{2\leq a\leq x^{\frac{1}{3}}} |\mu(a)| + O\left(x^{\frac{1}{4}}\cdot\ln x\right) \\
&= x - \frac{6}{\pi^2}\sqrt{x} - \frac{6}{\pi^2}x^{\frac{1}{3}} + O\left(x^{\frac{1}{4}}\cdot\ln x\right).
\end{aligned}$$

This completes the proof of Theorem.

§3. Some notes

If we use the deeply result for $\sum_{n\leq t}|\mu(n)|$, then we can get a sharper asymptotic formula for the mean value $\sum_{\substack{a\leq x\\ a\in A}}1$. For example, assume RH, then (See Theorem 3 of reference [4])

$$\sum_{n\leq t}|\mu(n)|=\frac{6}{\pi^2}t+O\left(x^{\frac{9}{28}+\epsilon}\right),$$

where ϵ be any fixed positive number.

Using this result we may immediately deduce that assume RH, then we have the asymptotic formula

$$\sum_{\substack{a\leq x\\ a\in A}}1=x-\frac{6}{\pi^2}\sqrt{x}-\frac{6}{\pi^2}x^{\frac{1}{3}}-\frac{6}{\pi^2}x^{\frac{1}{4}}-\frac{6}{\pi^2}x^{\frac{1}{5}}-\frac{6}{\pi^2}x^{\frac{1}{6}}+O\left(x^{\frac{9}{56}+\epsilon}\right),$$

where ϵ be any fixed positive number.

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Sub-self-conformal sets

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Abstract The definition of sub-self-conformal set is given, and we obtain the formula for their Hausdorff and box dimension.

Keywords Sub-self-conformal set, symbolic space, Hausdorff dimension, box dimension.

§1. Introduction

Self-similar sets presented by Hutchinson [1] have been extensively studied. See for example [2] [3] [5] [8]. Let's recall that. Let $X \subseteq \mathbf{R}^n$ be a nonempty compact set, and there exists $0 < C < 1$ such that

$$|w(x) - w(y)| \leq C|x - y|, \ \forall x, y \in X.$$

Then we say that $w : X \to X$ is a contractive map. If each $w_i (1 \leq i \leq m)$ is a contractive map from X to X , then we call $(X, \{w_i\}_{i=1}^m)$ the contractive iterated function systems(IFS). It is proved by Hutchinson that if $(X, \{w_i\}_{i=1}^m)$ is a contractive IFS, then there exists a unique nonempty compact set $E \subseteq \mathbf{R}^n$, such that

$$E = \bigcup_{i=1}^{m} w_i(E). \tag{1}$$

Set E is called a invariant set of IFS $\{w_i\}_{i=1}^m$. If each w_i is a contractive self-similar map, then we call $(X, \{w_i\}_{i=1}^m)$ the contractive self-similar IFS. Set E in (1) is a invariant set of w_i , and it is called a self-similar set. If each w_i is a contractive self-conformal map, then $(X, \{w_i\}_{i=1}^m)$ is called the contractive self-conformal IFS, and set E decided by (1) is called a self-conformal set.[4][6]

Recently, Falconer introduced sub-self-similar sets. Let w_i is self-similar IFS, and let F be a nonempty compact subset of $\mathbf{R}^n$ such that

$$F \subseteq \bigcup_{i=1}^{m} w_i(F).$$

This set F is called sub-self-similar set for w_i. Easy to see that, self-similar sets is a class of special sub-self-similar sets. At the same time, Falconer obtained the formula for the Hausdorff and box dimension of the sub-self-similar sets, when w_i satisfies the open set condition (OSC). In this paper, we will give the definition of sub-self-conformal sets similarly, and obtain the formula for their Hausdorff and box dimension.

At first, we give some definitions and remarks. By a symbolic space we mean the infinite product space $\Sigma = \{1, 2, \cdots, m\}^{\mathbf{N}}$, i.e. $\Sigma = \{(i_1, i_2, \cdots) : 1 \leq i_j \leq m$. Let $\Omega = \bigcup_{n\geq 0} \{1, 2, \cdots, m\}^n$. For $I = (i_1, i_2, \cdots, i_m) \in \Sigma$ and $k \in \mathbf{N}^+$, we write $I|_k = (i_1, i_2, \cdots, i_k)$, $I|^k = (i_{k+1}, i_{k+2}, \cdots)$, $\Sigma_k = \{I|_k : I \in \Sigma\}$. The shift transformation on Σ is defined by $\theta(I) = I|^1$. For $I = (i_1, i_2, \cdots), J = (j_1, j_2, \cdots) \in \Sigma$, define a metric d on Σ by

$$d(I, J) = e^{-k}, k = \max\{l : I|_l = J_l\}.$$

Obviously, if $I = J$, then $d(I, J) = 0$; and if $I_1 \neq J_1$, then $d(I, J) = 1$.

Let $\{w_i\}_{i=1}^m$ define as above, and B be any closed ball in $\mathbf{R}^n$ large enough to ensure that $w_i(B) \subseteq B, i = (1, 2, \cdots, m)$. Since each w_i is contracting, it is easy to see that the sequence of balls $w_{I|_k}(B) := w_{i_1} \circ w_{i_2} \circ \cdots \circ w_{i_k}(B)$ decreases with k and has intersection a single point. Then we let $\pi : \Sigma \to \mathbf{R}^n$ be defined by

$$\pi(I) = \bigcap_{k=1}^{\infty} w_{i_1} \circ w_{i_2} \ldots w_{i_k}(B).$$

Or

$$\pi(I) = \lim_{k\to\infty} w_{i_1} \circ w_{i_2} \ldots w_{i_k}(z), \ \forall z \in \mathbf{R}^n.$$

Definition 2.1. Suppose $(X, \{w_i\}_{i=1}^m)$ is a contractive self-conformal IFS, and F be a nonempty compact subset of X satisfies the condition

$$F \subseteq \bigcup_{i=1}^{m} w_i(F).$$

Then we call F a sub-self-conformal set for $\{w_i\}_{i=1}^m$.

§2. Symbolic space

For studying the Hausdorff and box dimension of the sub-self-conformal set, we set up a closed relationship between a sub-self-conformal set and a compact subset of the symbolic space.

Proposition 3.1.[4] Suppose π is defined as above, then the map $\pi : \Sigma \to \mathbf{R}^n$ is continuous. Similarly as the case of sub-self-similar set, we have this conclusion.

Proposition 3.2. F is a sub-self-conformal set for the IFS $\{w_i\}_{i=1}^m$ if and only if there is a set K, which is a shift invariant closed subset of Σ, such that $F = \pi(K)$.

Proof. Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$, then $F \subseteq \bigcup_{i=1}^{m} w_i(F)$. We define K as follows

$$K = \{(i_1, i_2, \cdots) : \pi(i_k, i_{k+1}, \cdots) \in F, \forall K \in \mathbf{Z}^+\}.$$

Obviously, for any $I \in K$, we have $\theta(I) \in K$. Then we will prove that $F = \pi(K)$. By the definition of K, we have $\pi(K) \subseteq F$. Since $F \subseteq \bigcup_{i=1}^{m} w_i(F)$, then for any $x_0 \in F$, we can find

some $x_1 \in F$ such that $x_0 = w_{i_1}(x_1)$, for some $i_1 : 1 \le i_1 \le m$. Similarly we have $x_1 = w_{i_2}(x_2)$ for some $x_2 \in F$ and $1 \le i_2 \le m$, and so on. We get a sequence $\{x_n\} \subset F$ and $i_n : 1 \le i_n \le m$ satisfy the condition

$$x_{k-1} = \lim_{n\to\infty} w_{i_k} \circ w_{i_{k+1}} \circ \cdots \circ w_{i_n}(x_n) = \bigcap_{j=k}^{\infty} w_{i_k} \circ \cdots \circ w_{i_j}(B) \in F, \forall k \in \mathbf{Z}^+.$$

So, $x_0 \in F$.

Conversely, let K be a compact subset of Σ, and $\theta(I) \in K$ for any $I \in K$. Suppose $F = \pi(K)$, then we will prove that $F \subseteq \bigcup_{i=1}^{m} w_i(F)$. In fact, for any $x \in \pi(K)$, there exist some $I = (i_1, i_2, \cdots) \in K$ such that

$$x = \pi(I) = \lim_{k\to\infty} w_{i_1} \circ \cdots \circ w_{i_k}(z), \quad \forall z \in \mathbf{R}^n.$$

Since $I|^1 = (i_2, i_3, \cdots) \in K$, then

$$x = w_{i_1}(\pi(I|^1) \in w_{i_1}(\pi(K)) \subseteq \bigcup_{i=1}^{m} w_i(\pi(K)),$$

so $\pi(K) \subseteq \bigcup_{i=1}^{m} w_i(\pi(K))$, i.e. F is a sub-self-conformal set for $\{w_i\}_{i=1}^m$.

From [8][9], we can find a similar proof.

Next, we will define a measure $M^s(A)$ on the symbolic space. By using this measure we can prove that: $\tau(s) = \lim_{k\to\infty} (\sum_{I\in K_k} |w_I'(x)|^s)^{\frac{1}{k}} = 1$, then

$$\dim_H F \le \underline{\dim}_B F \le \overline{\dim}_B F \le s.$$

Moreover, if $M^s(K) < \infty$, then $H^s(F) < \infty$, where $\pi(K) = F$. Finally we apply this theorem to attain the main conclusion of this paper: Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$ which satisfy the open set condition, then $s = \dim_H F = \dim_B F$, here s is the number satisfy

$$\tau(s) = \lim_{k\to\infty} (\sum_{I\in K_k} |w_I'(x)|^s)^{\frac{1}{k}} = 1.$$

§3. Hausdorff and box dimension of sub-self-conformal sets

To make the definition of sub-self-conformal sets plain or comprehensible, we give some examples.

e.g.1. Suppose $\{w_i\}_{i=1}^m$ is a self-conformal IFS and a nonempty compact set F satisfies $F = \bigcup_{i=1}^{m} w_i(F)$, here $m > 2$, i.e. F is an invariable set for $\{w_i\}_{i=1}^m$, then F is a sub-self-conformal set for $\{w_i\}_{i=1}^m$.

e.g.2. Let F be defined as e.g.1 and let ∂F be the boundary of F, then ∂F is a sub-self-conformal set for $\{w_i\}_{i=1}^m$.

Since for each $x \in \partial F$, we have $x \in F$. So there exist some $i : 1 \leq i \leq m$, such that $x \in w_i(F)$. At the same time, every neighborhood $\delta(x)$ of x contains points outside $w_i(F)$. i.e.

$$x \in \partial w_i(F) = w_i(\partial(F),$$

so

$$\partial F \subseteq w_i(\partial F) \subseteq \bigcup_{i=1}^{m} w_i(\partial F).$$

e.g.3. Let F be a sub-self-conformal set for $\{w_1, w_2, \cdots, w_m\}$, and E be a sub-self - conformal set for $\{s_1, s_2, \cdots, s_n\}$, then $F \cup E$ is a sub-self-conformal set for

$$\{w_1, w_2, \cdots, w_m, s_1, s_2, \cdots, s_n\}.$$

e.g.4. Let F be a sub-self-conformal set for $\{w_1, w_2, \cdots, w_m\}$, and E is a compact subset of F, and w is a contract self-conformal map, then $F \cup w(E)$ is a self-self-conformal set for $\{w_1, w_2, \cdots, w_m, w\}$.

Lemma 1.[6] Suppose X and $\{w_i\}_{i=1}^m$ be defined as above, and $\log |w_i'(x)|$ satisfy Dini condition, let $r_I = \inf_{x \in X} |w_I'(x)|, R_I = \sup_{x \in X} |w_I'(x)|$.

(i) There exist C_1 such that

$$R_I \leq C_1 r_I, \quad \text{for any } I \in \Omega.$$

(ii) There exist$C_2 > 0$ such that for any $x, y, z \in X, |x - y| \leq \delta$,

$$C_2^{-1} |w_I'(z)| \leq \frac{w_I(x) - w_I(y)}{x - y} \leq C_2 |w_I'(z), \quad \text{for any } I \in \Omega.$$

(iii) There exist $C_3 > 0$,such that for any $x, y \in X$,

$$|w_{IJ}'(x)| \leq C_3 |w_I'(x)||w_J'(x)| \quad \text{for any } I,\ J \in \Omega.$$

Next, we consider the dimension of the sub-self-conformal set. Let

$$K = \{(i_1, i_2, \cdots) : \pi(i_k, i_{k+1}, \cdots) \in F, \forall k \in \mathbf{Z}^+\} \subseteq \Sigma,$$

and K_k as above. For some $x \in F$, we write $r_I = |w_I'(x)|$, here $I = (i_1, i_2, \cdots, i_k)$. By lemma 1 (iii), we know that for any $I, J \in \Omega$ there exists C such that $r_{IJ} \leq C r_I r_J$, hence there exists C' such that

$$\sum_{I \in K_{k+l}} r_i^s \leq \sum_{I \in K_k \& J \in K_l} r_{IJ}^s \leq C' (\sum_{I \in K_k} r_I^s)(\sum_{I \in K_l} r_I^s).$$

Lemma 2.[3] Let $\{b_k\}_{k=1}^{\infty}$ be a positive real number sequence. Suppose for any positive integer $k,\ m$ we have $b_{k+m} \leq b_k + b_m$, then $\lim_{k \to \infty} \frac{b_k}{k}$ exist.

Lemma 3. The limit $\tau(s) := \lim_{k \to \infty} (\sum_{I \in \mathrm{K}_k} r_I^s)^{1/k}$ is exist.

Proof. By lemma 1(iii), we have

$$C_3 r_{IJ} \leq (C_3 r_I)(C_3 r_J). \tag{2}$$

We choose logarithm to (2), and let $b_k = \log \sum_{I\in K_k} r_I^s + s\log C_3$. Through lemma 2, we can prove that $\log(\sum_{I\in K_k} r_I^s)^{\frac{1}{k}} = \frac{b_k}{k}$ is convergent. Hence we know that $\lim_{k\to\infty}(\sum_{I\in K_k} r_I^s)^{1/k}$ exist.

Then we define a measure on the symbolic system. By using this measure we obtain the dimension of the sub-self-conformal set. We define

$$M_k^s(A) = \inf\left\{\sum_I |r_I|^s : A \subseteq \bigcup_I \sigma_I, |I| \geq k\right\},$$

here $A \subseteq \Sigma$, $s \geq 0, k = 1, 2, \cdots$, and $\sigma_I = \{IJ : J \in \Sigma\}$, for $I \in K_k$. Simility to the hausdorff measure we define

$$M^s(A) = \lim_{k\to\infty} M_k^s(A).$$

Propose 4.[10][11][12] Let A be a Borel set of Σ. If $0 < M^s(A) \leq \infty$, then there exists a compact subset $A_0 \subseteq A$ and a $b > 0$, such that $0 < M^s(A_0) < \infty$, it follows that

$$M^s(A_0 \cap \sigma_I) \leq br_I^s, \quad \forall I \in \Omega.$$

Propose 5. Let K, r_I be defined as above, then we have some conclusion
(i) There exists s_1 which satisfies $\tau(s_1) = 1$;
(ii) $s_2 := \inf\{s \geq 0 : M^s(K) = 0\} = \sup\{s \geq 0 : M^s(K) = \infty\}$;
(iii) $s_3 := \inf\left\{s \geq 0 : \sum_{k=1}^{\infty}\sum_{I\in K_k} r_I^s < \infty\right\} = \sup\left\{s \geq 0 : \sum_{k=1}^{\infty}\sum_{I\in K_k} r_I^s = \infty\right\}$.
At the same time we have $s_1 = s_2 = s_3$, and

$$M^{s_1}(K) \geq 1.$$

Proof. At first we will prove that s_1 is existent. Note that $r = \min_{1\leq i\leq m}\{r_i\}$, $R = \max_{1\leq i\leq m}\{r_i\}$. Then for $h > 0$, we have

$$r^{kh} \leq \frac{\sum_{I\in K_k} r_I^{s+h}}{\sum_{I\in K_k} r_I^s} \leq R^{kh}.$$

Let $k \to \infty$, it follows that

$$r^h \leq \frac{\tau(s_1 + h)}{\tau(s_1)} \leq R^h < 1,$$

$s_1 \geq 0, h > 0$.

Hence τ is continuous and strictly decreasing. Moreover, $\tau(0) \geq 1$ and $\tau(s_1) \to 0$ if $s_1 \to \infty$. So there is a unique $s_1 \geq 0$ satisfies $\tau(s_1) = 1$. By the standard feature of measure and progression [13][14], we know that

$$\inf\{s \geq 0 : M^s(K) = 0\} = \sup\{s \geq 0 : M^s(K) = \infty\}$$

and

$$\inf\left\{s \geq 0 : \sum_{k=1}^{\infty}\sum_{I\in K_k} r_I^s < \infty\right\} = \sup\left\{s \geq 0 : \sum_{k=1}^{\infty}\sum_{I\in K_k} r_I^s = \infty\right\}.$$

We write them s_2 and s_3 apartly. Next, we will prove that $s_1 \leq s_2$. Suppose $M^s(K) < 1$, for some $s > 0$. Here we have $s \geq s_2$. Then there is a covering $\bigcup_{I\in Q} \sigma_I$ of K such that $\sum_{I\in Q} r_I^s < 1$. Hence there exists $t : 0 < t < s$, such that $\sum_{I\in Q} r_I^t < 1$. Since K is compact, we can find a finite Q. Write $q = \max\{|I| : I \in Q\}$. Let

$$Q_k = \{I_1 I_2 \cdots I_p : I_j \in Q, \quad |I_1 I_2 \cdots I_{p-1}| \leq k < |I_1 I_2 \cdots I_p|\},$$

here $k \geq q$, and $|I|$ denote the length of I. Let K be a positive integer, we have $K \cap \sigma_I \subseteq \bigcup_{J\in Q} (K \cap \sigma_{IJ})$, for $I \in K_k$. Thus if $I \in K_k$, then $IJ \in Q_k$ for some $J \in Q$. Hence, we have that for each k

$$\sum_{I\in K_k} r_I^t \leq r^{-qt} \sum_{I'\in Q_k} r_{I'}^t \leq r^{-qt}.$$

Thus, if $M^s(K) < 1$, then for some $0 < t < s$ we have

$$M^t(K) \leq \lim_{k\to\infty} \sum_{I\in K_k} r_I^t \leq r^{-qt} < \infty.$$

So $\tau(s) < \tau(t) \leq 1$, i.e. $s \geq s_1$. It follows that $s_1 \leq s_2$. $s_2 \leq s_3$ is obviously. We cover K by cylinders σ_I, here $I \in \bigcup_{k=0}^{\infty} K_k$. Then if $\sum_{k=1}^{\infty} \sum_{I\in K_k} r_I^s < \infty$, we have $M_k^s(K) \leq \sum_{j=1}^{\infty} \sum_{I\in K_j} r_I^s$ for $k \to \infty$. Thus, if $s \geq s_3$, we have $s \geq s_2$.

To see that $s_1 = s_3$, we observe that $\sum_{k=1}^{\infty} \sum_{I\in K_k} r_I^{s_3}$ converges if $\tau(s_1) < 1$ and diverges if $\tau(s_1) < 1$.

In conclusion, we have s_1, s_2, s_3 exist and $s_1 = s_2 = s_3$.

On the side, we can know that if $M^{s_1}(K) < 1$, then $\tau(s_1) < 1$. From the proof of $s_1 \leq s_2$, we have that if $\tau(s_1) = 1$ then $M^{s_1}(K) \geq 1$.

Propose 6. Let F be a sub-self-conformal set, and $\tau(s) = 1$, then

$$\dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F \leq s.$$

Hence, if $M^s(K) < \infty$, then $H^s(F) < \infty$.

Proof. Suppose B is a closed ball such that $w_i(B) \subseteq B$, for any $i(1 \leq i \leq m)$. Let δ satisfy $0 \leq \delta \leq |B|$. For any $I = (i_1, i_2, \cdots) \in K$, we can find a corresponding $k \in \mathbf{Z}^+$ such that

$$r\delta < \left|w_{I|_k}(B)\right| = \left|w'_{I|_k}(\xi)\right| |B| \leq \delta, \quad \forall \xi \in B.$$

Write

$$Q_\delta = \{I \in K_k : r\delta < |w'_I(\xi)|\,|B| = |w_I(B)| \leq \delta\}.$$

We have that $F \subseteq \bigcup_{I\in Q_\delta} w_I(B)$ and $\bigcup_{I\in Q_\delta} w_I(B)$ is a cover of F. If N_δ is the number of sets in this cover, then for $t > s$,

$$N_\delta \delta^t = \sum_{I\in Q_\delta} \delta^t \leq r^{-t} |B|^t \sum_{I\in Q_\delta} |w'_I(\xi)| \leq C_2 r^{-t} |B|^t \sum_{k=1}^{\infty} \sum_{I\in K_k} r_I \leq M < \infty.$$

It follows that $\overline{\dim}_B \leq t$ for all $t > s$, and so $\overline{\dim}_B F \leq t$.

Since $K \subseteq \bigcup_{I\in Q} \sigma_I$, then $F \subseteq \bigcup_{I\in Q} w_I(B)$. So

$$H^s_\delta(F) \leq C_3 \left|B\right|^s M^s_k(K).$$

Here $\delta \geq r^k$. Thus if $M^s(K) < \infty$, then $H^s(F) < \infty$.

Lemma 7.[12] Let $\{V_i\}$ be a collection of disjoint open subsets of $\mathbf{R}^n$ such that each V_i contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$. Then any set U of diameter at most r intersects at most $b_1 := (1 + 2a_2)^n a_1^{-n}$ of the closures $\left\{\overline{V_i}\right\}$.

Theorem 8. Let F be a sub-self-conformal set for $\{w_i\}_{i=1}^m$ which satisfy the open set condition, and let $\tau(s) = 1$, then $H^s(F) > 0$ and

$$s = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F.$$

Proof. We can know that $M^s(K) > 0$ from propose 5. Let A_0 be a compact subset of K which confirm by propose 4. The Borel measure μ which supported by K is defined by

$$\mu : \mu(P) = M^s(A_0 \cap P),$$

where $P \subseteq \Sigma$. By the propose4 we have

$$\mu(\sigma_I) \leq b r_I^s, \quad \forall I \in K_k, k \in \mathbf{N}.$$

Let

$$\tilde{\mu}(U) = \mu\left\{J : \pi(J) \in U\right\}.$$

For $U \subset \mathbf{R}^n$. Such $\tilde{\mu}$ is a Borel measure supports by F. Let V be an open set satisfying OSC, and let $U \subset \mathbf{R}^n$ satisfy $0 < |U| < |V|$. Let

$$Q = \left\{(i_1, \cdots, i_k) : r_{i_1} r_{i_2} \cdots r_{i_k} \left|V\right| < \left|U\right| \leq r_{i_1} r_{i_2} \cdots r_{i_{k-1}} \left|V\right|\right\}.$$

For any fix $x \in V$. Since $rC_4 \left|U\right| \leq \left|w_I(V)\right| < C_4 \left|U\right|$ and using Lemma 7, we have that there are at most b_1 element in the family

$$Q_0 := \left\{I \in Q : U \cap w_I(\overline{V}) \neq \phi\right\}.$$

Thus, if $\pi(J) \in U$, then there exists $k \in \mathbf{N}$ such that $J|_k \in Q$, and so $J \in \sigma_I$ for some $I \in Q_0$. Hence

$$\tilde{\mu}(U) = \sum_{I\in Q_0} \mu\left\{J \in \sigma_I\right\} \leq b \sum_{I\in Q_0} r_I^s \leq b b_1 \left|V\right|^{-s} \left|U\right|^s.$$

Since $\tilde{\mu}$ is supported by F, the mass distribution principle implies that $H^s(F) > 0$. So $s = \dim_H F = \underline{\dim}_B F = \overline{\dim}_B F$, which s is the number satisfying $\tau(s) = 1$.

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On mean values of an arithmetic function[1]

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Abstract In this paper we shall study several kinds of mean values of the multiplicative arithmetic function $D(n)$ by the convolution method.

Keywords Arithmetic function, mean value, convolution method.

§1. Introduction and Results

We define the arithmetic function $D(n)$ by

$$D(n) = \begin{cases} 1, & \text{if } n = 1, \\ \prod\limits_{p^\alpha \| n} \alpha p^{\alpha-1}, & \text{if } n > 1. \end{cases}$$

In her doctoral thesis, Wang Xiaoying [3] proved the asymptotic formula

$$\sum_{n \le x} \frac{1}{D(n)} = c_1 x + O(x^{\frac{1}{2}+\varepsilon}), \tag{1}$$

where

$$c_1 := \frac{6}{\pi^2} \prod_p \left(1 - \frac{p^2}{p+1}\left(\log\left(1 - \frac{1}{p^2}\right) + \frac{1}{p^2}\right)\right).$$

In this short paper we first show that the asymptotic formula (1) can be slightly improved by the following

Theorem 1. There exists an absolute constant $c > 0$, such that the asymptotic formula

$$\sum_{n \le x} \frac{1}{D(n)} = c_1 x + O(x^{\frac{1}{2}} e^{-c\delta(x)}) \tag{2}$$

is true, where $\delta(x) := \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}$.

Remark. It is very difficult to improve the exponent 1/2 in (1) and (2), unless we have substantial progress in the study of the zero region of $\zeta(s)$. Therefore it is reasonable to study the problem in short intervals. In this case we have the following

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Theorem 2. If $x^{\frac{1}{5}+3\varepsilon} \le y \le x$, then

$$\sum_{x<n\le x+y} \frac{1}{D(n)} = c_1 y + O(yx^{-\varepsilon}). \tag{3}$$

Finally we prove the following Theorem 3, which studies mean values of the function $\log D(n)$.

Theorem 3. We have

$$\sum_{n\le x} \log D(n) = c_2 x + O(x^{\frac{1}{2}+\varepsilon}), \tag{4}$$

where

$$c_2 := \sum_p \left(1-\frac{1}{p}\right) \sum_{n=2}^{\infty} p^{-n} \log(np^{n-1}).$$

Notations. Throughout this paper, $\varepsilon > 0$ denotes a small positive constant, $\mu(n)$ denotes the Möbius function, $\delta(x) := \log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}}$.

§2. Proof of the theorems

We first prove Theorem 1. Let $f(s) = \sum_{n=1}^{\infty} \frac{1}{D(n)n^s}$ $(Res > 1)$. By the Euler product we get

$$\begin{aligned} f(s) &= \prod_p \left(1+\frac{1}{p^s}+\frac{1}{2p^{2s+1}}+\cdots+\frac{1}{np^{ns+n-1}}+\cdots\right) \\ &= \prod_p \left(1+\frac{1}{p^s}\right)\left[1-\frac{p}{1+\frac{1}{p^s}}\left(\log\left(1-\frac{1}{p^{s+1}}\right)+\frac{1}{p^{s+1}}\right)\right] \\ &= \frac{\zeta(s)}{\zeta(2s)}G(s), \end{aligned} \tag{5}$$

where $G(s) = \prod_p \left[1-\frac{p}{1+\frac{1}{p^s}}\left(\log\left(1-\frac{1}{p^{s+1}}\right)+\frac{1}{p^{s+1}}\right)\right]$.

Let $G(s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$. It is easy to prove that the series is absolutely convergent for $Res > \varepsilon > 0$. So we have

$$\sum_{n\le x} |\, g(n)\,| \ll x^{\epsilon}. \tag{6}$$

From (5) we see that the relation

$$\frac{1}{D(n)} = \sum_{dl=n} \mu^2(d) g(l) \tag{7}$$

holds.

We have the well-known asymptotic formula

$$\sum_{n\le x} \mu^2(d) = \frac{6}{\pi^2}x + O(x^{\frac{1}{2}} e^{-c_0\delta(x)}), \tag{8}$$

where $c_0 > 0$ is an absolute constant.

Now Theorem 1 follows from (6)-(8) and Theorem 14.2 of Ivić[2].

In order to prove Theorem 2, we need the following result(see [1]): If $x^{\frac{1}{5}+2\epsilon} \le y \le x$, then we have

$$\sum_{x<d\le x+y} \mu^2(d) = \frac{6}{\pi^2} y + O(yx^{-\epsilon}). \tag{9}$$

Suppose $x^{\frac{1}{5}+3\varepsilon} \le y \le x$ and let $M = x^{2\varepsilon}$. For $l \le M$, we have $\left(\frac{x}{l}\right)^{\frac{1}{5}+\epsilon} \le \frac{y}{l} \le \frac{x}{l}$. By (7), we get

$$\begin{aligned}
\sum_{x<n\le x+y} \frac{1}{D(n)} &= \sum_{x<dl\le x+y} \mu^2(d) g(l) \\
&= \sum_{l\le x+y} g(l) \sum_{\frac{x}{l}<d\le\frac{x+y}{l}} \mu^2(d) \\
&= \sum_{l\le M} g(l) \sum_{\frac{x}{l}<d\le\frac{x}{l}+\frac{y}{l}} \mu^2(d) + \sum_{M<l\le x+y} g(l) \sum_{\frac{x}{l}<d\le\frac{x}{l}+\frac{y}{l}} \mu^2(d) \\
&= \sum_1 + \sum_2
\end{aligned} \tag{10}$$

say.

Using (9), (10) and partial summation, we get

$$\begin{aligned}
\sum_1 &= \sum_{l\le M} g(l) \left(\frac{6}{\pi^2} \frac{y}{l} + O\left(\frac{y}{l} \left(\frac{x}{l} \right)^{-\epsilon} \right) \right) \\
&= \frac{6}{\pi^2} G(1) y + O\left(y \sum_{l>M} \frac{|\, g(l)\,|}{l} \right) + O\left(yx^{-\epsilon} \sum_{l\le M} \frac{|\, g(l)\,|}{l^{1-\epsilon}} \right) \\
&= c_1 y + O(yx^{-\epsilon}),
\end{aligned} \tag{11}$$

$$\sum_2 \ll \sum_{M<l\le x+y} |g(l)|(y/l+1) \ll yx^{-\varepsilon}. \tag{12}$$

Theorem 2 follows from (10)-(12).

Finally we prove Theorem 3. Suppose λ is a complex number with $-\varepsilon < \Re\lambda < \varepsilon$. Consider $F(s,\lambda) = \sum_{n=1}^{\infty} \frac{1}{D(n)^\lambda n^s}$ $(\Re s > 1)$. By Euler product, we have

$$\begin{aligned}
F(s,\lambda) &= \prod_p \left(1 + \frac{1}{D^\lambda(p)p^s} + \frac{1}{D^\lambda(p^2)p^{2s}} + \cdots + \frac{1}{D^\lambda(p^n)p^{ns}} \cdots \right) \\
&= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{2^\lambda p^{2s+\lambda}} + \frac{1}{3^\lambda p^{3s+2\lambda}} + \cdots + \frac{1}{n^\lambda p^{ns+(n-1)\lambda}} + \cdots \right) \\
&= \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \prod_p \left(1 - \frac{1}{p^s} \right) \left(1 + \frac{1}{p^s} + \frac{1}{2^\lambda p^{2s+\lambda}} + \cdots + \frac{1}{n^\lambda p^{ns+(n-1)\lambda}} + \cdots \right) \\
&= \zeta(s) K(s,\lambda),
\end{aligned} \tag{13}$$

where

$$K(s,\lambda)=\prod_p\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{p^s}+\frac{1}{2^\lambda p^{2s+\lambda}}+\cdots+\frac{1}{n^\lambda p^{ns+(n-1)\lambda}}+\cdots\right). \tag{14}$$

Let $K(s,\lambda)=\sum_{n=1}^{\infty}\frac{k(n,\lambda)}{n^s}$. It is easily to see that the series is absolutely convergent for $\Re s>\frac{1}{2}+\frac{\Re\lambda}{2}+\frac{\epsilon}{2}$. So we have

$$\sum_{n\le x}k(n,\lambda)\ll x^{1/2+\Re\lambda/2+\varepsilon}. \tag{15}$$

From (13)-(14) and Theorem 14.1 of Ivić[2], we get

$$\sum_{n\le x}D^{-\lambda}(n)=h(\lambda)x+O(x^{1/2+\Re\lambda/2+\varepsilon}), \tag{16}$$

where $h(\lambda)=K(1,\lambda)$.

Taking the derivative for λ from both sides of (16), and then letting $\lambda\to 0$, we get

$$\sum_{n\le x}\log D(n)=-h'(0)x^{1/2}+O(x^{1/2+\varepsilon}). \tag{17}$$

It is easy to check that

$$h'(0)=-\sum_p\left(1-\frac{1}{p}\right)\sum_{n=2}^{\infty}p^{-n}\log\left(np^{n-1}\right).$$

This completes the proof of Theorem 3.

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On an infinite series related to Hexagon-numbers[1]

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Abstract Let $a(n)$ denotes the Hexagon-number part of n. In this short paper, we shall show that the infinite series $\sum_{n=1}^{\infty}\frac{1}{a^s(n)}(\Re s > 1)$ can be continued memomorphically to the whole complex plane with simple poles $1,\ 1/2,\ -1/2,\ -3/2,\ \cdots$.

Keywords Hexagon-number, Abel's summation formula, the analytic continuation.

§1. Introduction

For any positive integer m, if $n = m(2m-1)$, then we say n is a Hexagon-number(see reference [1]). For any positive integer n, let m be the largest positive integer which satisfies the inequality

$$m(2m-1) \le n < (m+1)(2m+1).$$

Define $a(n) := m(2m-1)$, and call $a(n)$ the Hexagon-number part of n. The second-named author[1] proved that for any real $s > 1$ the infinite series $f(s) = \sum_{n=1}^{\infty}\frac{1}{a^s(n)}$ is convergent and that $f(2) = \frac{5}{3}\pi^2 - 4\ln 2$.

In this short paper, we shall prove the following

Theorem. The Dirichlet series $f(s) = \sum_{n=1}^{\infty}\frac{1}{a^s(n)}$ can be continued to the whole complex plane as a meromorphic function with simple poles $1,\ 1/2,\ -1/2,\ -3/2,\ -5/2,\ \cdots$.

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§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. First let

$$b(n) := \begin{cases} 4m+1, & \text{if } n = m(2m-1), \\ 0, & \text{otherwise.} \end{cases}$$

Then, if $\Re s = \sigma > 1$, we have(see reference[1])

$$f(s) = \sum_{n=1}^{\infty} \frac{1}{a^s(n)} = \sum_{n=1}^{\infty} \frac{4m+1}{m^s(2m-1)^s} = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

For any real u, let $[u]$ denote the greatest integer not exceeding u, $b_1(u) = u - [u] - \frac{1}{2}$. We define a series of functions $b_k(u)$ $(k \geq 2)$ by the following relation:

$$b_{k+1}(u) - b_{k+1}(0) = \int_0^u b_k(t)dt, \quad \int_0^1 b_{k+1}(t)dt = 0, k = 1, 2, \cdots .$$

Then for each $k \geq 1$ the function the $b_k(u)$ is a periodic function with period 1. Especially we have (see [2])

$$b_2(u) = \frac{1}{2}(u - [u])^2 - \frac{1}{2}(u - [u]) + \frac{1}{12}.$$

For any $u \geq 1$, define $B(u) := \sum_{n \leq u} b(n)$, $u_0 := \frac{1}{4} + \sqrt{\frac{u}{2} + \frac{1}{16}}$. Then

$$\begin{aligned} B(u) &= \sum_{m(2m-1) \leq u} (4m+1) \\ &= \sum_{m \leq u_0} (4m+1) \\ &= 2[u_0]([u_0]+1) + [u_0] \\ &= 2\left[(u_0 - b_1(u_0))^2 - \frac{1}{4}\right] + u_0 - b_1(u_0) - \frac{1}{2} \\ &= 2u_0^2 + u_0 - \frac{5}{6} + 4b_2(u_0) - 4u_0 b_1(u_0) - b_1(u_0) \\ &= B_1(u) + B_2(u), \end{aligned}$$

where

$$\begin{aligned} B_1(u) &:= 2u_0^2 + u_0 - \frac{5}{6}, \\ B_2(u) &:= 4b_2(u_0) - 4u_0 b_1(u_0) - b_1(u_0). \end{aligned}$$

Suppose $x \geq 1$ is a fixed real number, we write

$$\begin{aligned} f(s) &= \sum_{n \leq x} \frac{b(n)}{n^s} + \sum_{n > x} \frac{b(n)}{n^s} \\ &= \sum\nolimits_1 + \sum\nolimits_2 . \end{aligned} \tag{1}$$

Using Abel's summation formula, we have

$$\begin{aligned}\sum\nolimits_2 &= \sum_{n>x}\frac{b(n)}{n^s} = \int_x^\infty \frac{1}{u^s}dB(u)\\ &= \int_x^\infty \frac{1}{u^s}dB_1(u) + \int_x^\infty \frac{1}{u^s}dB_2(u). \qquad (2)\end{aligned}$$

Now we evaluate the integrals $\int_x^\infty \frac{1}{u^s}dB_1(u)$ and $\int_x^\infty \frac{1}{u^s}dB_2(u)$, respectively. Our main tool is partial summation and the power series.

$$\begin{aligned}\int_x^\infty \frac{dB_1(u)}{u^s} &= \int_x^\infty \frac{1+\frac{2}{\sqrt{8u+1}}}{u^s}du\\ &= \int_x^\infty \frac{du}{u^s} + 2\int_x^\infty \frac{du}{u^s\sqrt{8u+1}}\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} 2\int_x^M \frac{du}{u^s\sqrt{8u+1}}\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} 2\int_x^M \frac{du}{u^s(8u)^{\frac12}(1+\frac{1}{8u})^{\frac12}}\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}\int_x^M \frac{du}{u^{s+\frac12}(1+\frac{1}{8u})^{\frac12}}\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}\int_x^M \frac{1}{u^{s+\frac12}}\sum_{n=0}^\infty(-1)^n\frac{(2n-1)!!}{(2n)!!}(\frac{1}{8u})^n du\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}\sum_{n=0}^\infty(-\frac18)^n\frac{(2n-1)!!}{(2n)!!}\int_x^M \frac{1}{u^{n+s+\frac12}}du\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}\sum_{n=0}^\infty(-\frac18)^n\frac{(2n-1)!!}{(2n)!!}\frac{1}{\frac12-s-n}(M^{\frac12-s-n}-x^{\frac12-s-n})\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}\sum_{n=0}^\infty(-\frac18)^n\frac{(2n-1)!!}{(2n)!!}\frac{M^{\frac12-s-n}}{\frac12-s-n}\\ &-\frac{1}{\sqrt2}\sum_{n=0}^\infty(-\frac18)^n\frac{(2n-1)!!}{(2n)!!}\frac{x^{\frac12-s-n}}{\frac12-s-n}\\ &= \frac{x^{1-s}}{s-1} + \lim_{M\to\infty} \frac{1}{\sqrt2}M^{\frac12-s}\sum_{n=0}^\infty(-1)^n\frac{(2n-1)!!}{(2n)!!}(\frac{1}{8M})^n\frac{1}{\frac12-s-n}\\ &-\frac{1}{\sqrt2}\sum_{n=0}^\infty(-\frac18)^n\frac{(2n-1)!!}{(2n)!!}\frac{x^{\frac12-s-n}}{\frac12-s-n}.\end{aligned}$$

Recall $\Re s = \sigma > 1$, so

$$\sum_{n=0}^\infty(-1)^n\frac{(2n-1)!!}{(2n)!!}(\frac{1}{8M})^n\frac{1}{\frac12-s-n} \ll \sum_{n=0}^\infty(\frac{1}{8M})^n \ll 1,$$

and correpondingly

$$M^{\frac12-s}\sum_{n=0}^\infty(-1)^n\frac{(2n-1)!!}{(2n)!!}(\frac{1}{8M})^n\frac{1}{\frac12-s-n} \ll M^{\frac12-\sigma}.$$

Thus we get($\sigma > 1$),

$$\int_x^\infty \frac{dB_1(u)}{u^s} = \frac{x^{1-s}}{s-1} - \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty}(-\frac{1}{8})^n \frac{(2n-1)!!}{(2n)!!}\frac{x^{\frac{1}{2}-s-n}}{\frac{1}{2}-s-n} = f_x(s), \tag{3}$$

say. It is easy to see that the power series in (3) is absolutely and uniformly convergent in any compact domain in $\mathbb{C} \setminus \{1/2,\ -1/2,\ -3/2,\ -5/2,\ \cdots\}$. So $f_x(s)$ represents a meromorphic function on $\mathbb{C}$ with simple poles $1,\ 1/2,\ -1/2,\ -3/2,\ -5/2,\ \cdots$.

Now consider $\int_x^\infty \frac{dB_2(u)}{u^s}$. By partial integral, we have

$$\begin{aligned}\int_x^\infty \frac{dB_2(u)}{u^s} &= \frac{B_2(u)}{u^s}|_x^\infty + s\int_x^\infty \frac{B_2(u)}{u^{s+1}}du \\ &= -\frac{B_2(x)}{x^s} + 4s\int_x^\infty \frac{b_2(u_0)}{u^{s+1}}du - 4s\int_x^\infty \frac{u_0 b_1(u_0)}{u^{s+1}}du - s\int_x^\infty \frac{b_1(u_0)}{u^{s+1}}du \\ &= -\frac{B_2(x)}{x^s} + 4s\int_1 - 4s\int_2 - s\int_3. \end{aligned} \tag{4}$$

We study only $\int_1$. Let $g_1(v) = \frac{4v-1}{(2v^2-v)^{s+1}}$, $x_0 = \frac{1}{4} + \sqrt{\frac{x}{2} + \frac{1}{16}}$, and $k \geq 1$ be an integer. By the change of variable $v = u_0$ and repeated partial integration we get

$$\begin{aligned}\int_1 &= \int_{x_0}^\infty \frac{b_2(v)(4v-1)}{(2v^2-v)^{s+1}}dv \\ &= \int_{x_0}^\infty b_2(v)g_1(v)dv \\ &= g_1(v)b_3(v)|x_0^\infty - \int_{x_0}^\infty g_1'(v)b_3(v)dv \\ &= -g_1(x_0)b_3(x_0) - \int_{x_0}^\infty g_1'(v)db_4(v) \\ &= \cdots\cdots \\ &= -g_1(x_0)b_3(x_0) + g_1'(x_0)b_4(x_0) - g_1''(x_0)b_5(x_0) + \cdots\cdots + (-1)^k g_1^{(k-1)}(x_0)b_{k+2}(x_0) \\ &+(-1)^k \int_{x_0}^\infty b_{k+2}(v)g_1^{(k)}(v)dv. \end{aligned} \tag{5}$$

For $v \geq 1$, we have the power series expansion

$$\begin{aligned} g_1(v) &= (4v-1)(2v^2)^{-s-1}\left(1 - \frac{1}{2v}\right)^{-s-1} \\ &= (4v-1)(2v^2)^{-s-1}\sum_{n=0}^{\infty}\frac{(s+1)(s+2)\cdots(s+n)}{n!}\left(\frac{1}{2v}\right)^n \\ &= (4v-1)\sum_{n=0}^{\infty}\frac{(s+1)(s+2)\cdots(s+n)}{2^{s+n+1}n!}\left(\frac{1}{v}\right)^{2s+n+2}, \end{aligned}$$

which implies that $g_1^{(k)}(v) \ll v^{-2\sigma-k-1}$. Thus the integral $\int_1$ is absolulely convergent for $\sigma > -\frac{k}{2}$ and represents an analytic function in this half plane. Similarly, the $\int_2$ and $\int_3$ are

analytic in the range $\sigma > -k/2$, too. It is easy to check that the other terms in (5) form an entire function in the whole plane. From the above we see that $\int_x^\infty \frac{dB_2(u)}{u^s}$ has an analytic contiuation to $\sigma > -\frac{k}{2}$. Since k is arbitrary, the integral $\int_x^\infty \frac{dB_2(u)}{u^s}$ has an analytic contiuation to $\mathbb{C}$.

Combing(1), (2), (3), (4) and (5) completes the proof of Theorem.

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A structure theorem of right C-rpp semigroups[1]

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Abstract A new method of construction for right C-rpp semigroups is given by using a right cross product of a right regular band and a strong semilattice of left cancellative monoids.

Keywords Right C-rpp semigroups, right cross products, right regular bands, left cancellative monoids.

§1. Introduction

Recall that a semigroup S is called an rpp semigroup if all its principal right ideals aS^1, regarded as right S^1-systems, are projective.

According to J.B. Fountain[5], a semigroup S is rpp if and only if, for any $a \in S$, the set

$$M_a=\{e \in E \mid S^1a \subseteq Se \text{ and for all } x, y \in S^1,\ ax = ay \Rightarrow ex = ey\}$$

is a non-empty set, where E is the set of all idempotents of S. An rpp semigroup S is called strongly rpp if for every $a \in S$, there exists a unique idempotent e in M_a such that ea=a. It is easy to see that regular semigroups are rpp semigroups and completely regular semigroups are strongly rpp semigroups. Thus, rpp semigroups are generalizations of regular semigroups. A strongly rpp semigroups S is said to be a right C-rpp semigroup if $\mathcal{L}^* \vee \mathcal{R}$ is a congruence on S and $Se \subseteq eS$ for all $e \in E(S)$.

It is clear that a right C-rpp semigroup is a generalization of a right inverse semigroup (see [8]). Right C-rpp semigroups have been investigated by Guo and Shum-Ren in [3] and [2].

In this paper, we will give another construction of such semigroups by using right cross product of semigroups.

§2. Right cross product structure

We first introduce the concept of right cross products of semigroups.

Let Y be a semilattice. Let $M = [Y; M_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of left cancellative monoids M_α with structure homomorphism $\varphi_{\alpha,\beta}$ and $\Lambda = \cup_{\alpha \in Y}\Lambda_\alpha$ be a semilattice decomposition of a right regular band Λ into right zero bands Λ_α.

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Denote the direct product of a cancellative monoids M_α and a right zero band Λ_α by $S_\alpha = M_\alpha \times \Lambda_\alpha$.

For any $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$, we define a mapping $\theta_{\alpha,\beta}$: $M_\alpha \times \Lambda_\beta \to \Lambda_\beta$ by $(a, j) \mapsto [a, j]$ satisfying the following conditions:

(i) If $\alpha = \beta$, then for any $a \in M_\alpha$, $j \in \Lambda_\beta$, $[a, j]$ is constant;

(ii) If $(a, i) \in S_\alpha$, $(b, j) \in S_\beta$, $k \in \Lambda_\gamma$, then $[ab, [a, ij]ik][a, ij]i = [a, i[b, jk]j]i$;

(iii) If $(a, i) \in S_\alpha$, $j \in \Lambda_\beta$, $k \in \Lambda_\gamma$ and $[a, ij]i = [a, ik]i$, then $[1_\alpha, ij]i = [1_\alpha, ik]i$.

Then, consider the set $S = \cup_{\alpha\in Y} S_\alpha$ with a binary operation "$\circ$": for any $(a, i) \in S_\alpha$, $(b, j) \in S_\beta$,

$$(a, i) \circ (b, j) = (ab, [a, ij]i),$$

where ab and ij are the usual semigroup products of a, b and i, j respectively in the semigroups M and Λ. Using conditions (ii) and (iii), we can verify that $(S, \circ)$ is a semigroup. Denote this semigroup by $S = M \rtimes_\theta \Lambda$ and call it a right cross product of M and Λ.

To give a new construction of a right *C-rpp* semigroup, we list a description for such semigroups.

Lemma 2.1.[2] Let S be a *rpp* semigroup. Then the following statements are equivalent:

(i) S is a right C-rpp semigroup;

(ii) S is a strongly rpp-semigroup, $\mathcal{L}^* \vee \mathcal{R}$ is a congruence on S and $Se \subseteq eS$ for all $e \in E(S)$;

(iii) S is a semilattice of direct products $M_\alpha \times \Lambda_\alpha$, where M_α is a left cancellative monoid and Λ_α is a right zero band for every $\alpha \in Y$;

(iv) S is a strongly rpp semigroup such that $\mathcal{D}^{(\ell)}$ is a semilattice congruence on S, and $\mathcal{D} \mid_{RegS} = \mathcal{R} \mid_{RegS}$, where Reg S is the set of all regular elements of S;

(v) S is semilattice of $\mathcal{D}^{(\ell)}$-simple strongly rpp semigroups, and $\mathcal{D} \mid_{RegS} = \mathcal{R} \mid_{RegS}$.

Now we are ready to establish a construction theorem for right *C-rpp* semigroups.

Theorem 2.2. Let Y be a semilattice. Let $M = [Y; M_a, \varphi_{\alpha,\beta}]$ is a strong semilattice of left cancellative monoids M_α with structure homomorphism $\varphi_{\alpha,\beta}$ and $\Lambda = \cup_{\alpha\in Y} \Lambda_\alpha$ be a semilattice decomposition of right regular band Λ into right zero bands Λ_α. Then, a right cross product $M \rtimes_\theta \Lambda$ of M and Λ is a right *C-rpp* semigroup.

Conversely, any right *C-rpp* semigroups can be constructed in this way.

Proof. To prove the direct part of Theorem 2.2, we first consider the set of idempotents of the semigroup $M \rtimes_\theta \Lambda$. Let $E = \cup_{\alpha\in Y}\{(1_\alpha, i) \mid i \in \Lambda_\alpha, 1_\alpha \text{ is the identity of } M_\alpha\}$. Clearly, for any $(1_\alpha, i)$, $(1_\alpha, i) \circ (1_\alpha, i) = (1_\alpha, [1_\alpha, ii]i) = (1_\alpha, i)$.

Conversely, if $(a, i)^2 = (a, i) \in M_\alpha \times \Lambda_\alpha$ for $\alpha \in Y$, then it is trivial to check that $a = 1_\alpha$, where 1_α is the identity of M_α. This shows that E is the set of all idempotents of $M \rtimes_\theta \Lambda$.

It is easy to see that $M \rtimes_\theta \Lambda$ is a semilattice of direct products $M_\alpha \times \Lambda_\alpha$.

To show that $M \rtimes_\theta \Lambda$ is a right *C-rpp* semigroup, by Lemma 2.1, we only need to prove that the semigroup $M \rtimes_\theta \Lambda$ is an *rpp* semigroup.

Suppose that $x \in M \rtimes_\theta \Lambda$. Then there exists $\alpha \in Y$ such that $x = (a, i) \in S_\alpha = M_\alpha \times \Lambda_\alpha$. Hence, for any $u = (b, j) \in {S_\beta}^1$, $v = (c, k) \in {S_\gamma}^1$, if $x \circ u = x \circ v$, that is, $(a, i) \circ (b, j) = (a, i) \circ (c, k)$, then $ab = ac$ clearly, and $[a, ij]i = [a, ik]i$. Since M is a strongly semilattice of left cancellative monoids M_α, we have that $b = c$.

Let $e = (1_\alpha, i) \in E$. By the condition (iii), we have $[1_\alpha, ij]i = [1_\alpha, ik]i$ and $e \circ u = e \circ v$. Again since $(a, i) \circ (1_\alpha, i)=(a, [a, i]i) = (a, i)$, that is, $x = x \circ e$, then we have $S^1x \subseteq S^1e$. Thus $M_x \neq \varnothing$. By the definition of an *rpp* semigroup, $x \in M \rtimes_\theta \Lambda$ is an *rpp* semigroup and so it is a right *C-rpp* semigroup.

We now begin to show how a right *C-rpp* semigroup S become a right cross product of M and Λ, where M is a strongly semilattice of left cancellative monoids and Λ is a right regular band. Our proof is divided the following steps:

(I) Assume that S is a right *C-rpp* semigroup. By Lemma 2.1, S can be written as a semilattice $\cup_{\alpha\in Y}(M_\alpha \times \Lambda_\alpha)$ of $M_\alpha \times \Lambda_\alpha$, where M_α is a left cancellative monoid and Λ_α is a right zero band. Denote $S_\alpha = M_\alpha \times \Lambda_\alpha$.

Now we let $M = \cup_{\alpha\in Y} M_\alpha$ and $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$.

Firstly, we will show that M is a strongly semilattice of M_α. For this purpose, let $E = \cup_{\alpha\in Y}\{(1_\alpha, i) \mid i \in \Lambda_\alpha,\ 1_\alpha \text{ is the identity of } M_\alpha\}$. Then it can be easily checked that E is the set of idempotents of S.

Now, for any $\alpha, \beta \in Y$ with $\alpha \geqslant \beta$, we let$(a, i) \in S_\alpha$, $(1_\beta, j) \in S_\beta \cap E$. Clearly,

$$(a, i)(1_\beta, j) = (b, k) \in S_\beta,$$

for some $b \in M_\beta$ and $k \in \Lambda_\beta$. But we have that $(a, i)(1_\beta, j)(1_\beta, j) = (b, k)(1_\beta, j) \in S_\beta$ and thereby

$$(a, i)(1_\beta, j) = (b, j) \in S_\beta.$$

This implies that the choice of b is independent on j. Moreover, since E is a right regular band, we see that for any $l \in \Lambda_\alpha$, $(a, l)(1_\beta, j) = (a, i)(1_\alpha, l)(1_\beta, j) = (a, i)[(1_\alpha, l)(1_\beta, j)](1_\beta, j) = (a, i)(1_\beta, j)$. This also shows that the choice of b is independent on i.

Consequently, if we define a mapping $\varphi_{\alpha,\beta} : M_\alpha \to M_\beta$ by $a \mapsto a\varphi_{\alpha,\beta}$ if and only if

$$(a, i)(1_\beta, j) = (b, k) = (a\varphi_{\alpha,\beta}, j).$$

Then the mapping $\varphi_{\alpha,\beta}$ is well define. By routine checking, all the mappings $\varphi_{\alpha,\beta}$ are indeed the structure homomorphisms of a strong semilattice Y of monoids. Thus, $M = [Y; M_a, \varphi_{\alpha,\beta}]$ is a strong semilattice of M_α.

Next, we claim that Λ is a right regular band. For this purpose, it suffice to show that Λ is isomorphic to $E(S)$ which is the set of all idempotents of S. Since S is a right *C-rpp* semigroup, the mapping $\omega : E(S) \to \Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$ given by $(1_\alpha, i) \mapsto i$ for $i \in \Lambda_\alpha$ is clearly bijective. We now define $ij = k$ for $i \in \Lambda_\alpha, j \in \Lambda_\beta$ if and only if

$$(1_\alpha, i)(1_\beta, j) = (1_{\alpha\beta}, k).$$

Then, we can easily see that the set $\Lambda = \cup_{\alpha\in Y}\Lambda_\alpha$, under the above multiplication, is isomorphic to $E(S)$ which is a right regular band as mentioned above.

(II) Now we consider a construction for structure mapping $\theta_{\alpha,\beta}$ of a right cross product $M \rtimes_\theta \Lambda$.

For any $\alpha, \beta \in Y$, with $\alpha \geq \beta$, let $(1_\beta, j) \in S_\beta \cap E$, and $(a, i^\circ) \in S_\alpha$, where i° is a fixed element. Then by Lemma 2.1, we have

$$(1_\beta, j)(a, i^\circ) = (a', k) \in S_\beta$$

and

$$\begin{aligned}(1_\beta, j)(a, i^\circ)(1_\beta, k) &= (1_\beta, j)(a, i^\circ)(1_\beta, k^\circ)(1_\beta, k)\\ &= (1_\beta, j)(a\varphi_{\alpha,\beta}, k^\circ)(1_\beta, k)\\ &= (a\varphi_{\alpha,\beta}, k).\end{aligned}$$

Consequently,

$$(1_\beta, j)(a, i^\circ) = (a\varphi_{\alpha,\beta}, k).$$

Thus, we may define a mapping $\theta_{\alpha,\beta}$: $M_\alpha \times \Lambda_\beta \to \Lambda_\beta$ by $(a, j) \mapsto [a, j] \in \Lambda_\beta$ if and only if $(1_\beta, j)(a, i^\circ) = (a\varphi_{\alpha,\beta}, [a, j])$. By using the above result, we have

$$\begin{aligned}(1_\beta, j)(a, i) &= (1_\beta, j)(a, i^\circ)(1_\alpha, i)\\ &= (a\varphi_{\alpha,\beta}, [a, j])(1_\beta, [a, j])(1_\alpha, i)\\ &= (a\varphi_{\alpha,\beta}, [a, j])(1_\beta, [a, j]i)\\ &= (a\varphi_{\alpha,\beta}, [a, j]i).\end{aligned}$$

It is easy to see that, if $\alpha = \beta$ for any $a \in M_\alpha, j \in 1_\alpha, [a, j] = i^\circ$. This shows that the condition (i) is satisfied by mapping $\theta_{\alpha,\beta}$.

(III) We will verify that the conditions (ii) and (iii) in the right cross product of M and Λ are satisfied by the mapping $\theta_{\alpha,\beta}$.

It follows by associativity of the semigroup product that the mapping $\theta_{\alpha,\beta}$ satisfies the condition (ii).

Next, we will show that the mapping $\theta_{\alpha,\beta}$ satisfies the condition (iii).

Since a right *C-rpp* semigroup is an *rpp* semigroup, we know that for any $x = (a, i) \in S$, $M_x = \{e \in E \mid S^1a \subseteq S^1e$ and for all $u, v \in S^1, xu = xv \Rightarrow eu = ev\}$ is a non-empty set. There exists an idempotent $e = (1_\alpha, i) \in M_x$ such that for any $u = (b, j), v = (c, k) \in S^1$, if $xu = xv$, then $eu = ev$. From this, we can easily verify that the condition (iii) is satisfied by the mapping $\theta_{\alpha,\beta}$.

(IV) It remains to show that the multiplication on the right C-rpp semigroup S coincides with the multiplication on the right cross product $M \rtimes_\theta \Lambda$ and so $S \cong M \rtimes_\theta \Lambda$.

For any $(a, i) \in S_\alpha, (b, j) \in S_\beta, \alpha, \beta \in Y$, by using the above result, we have

$$\begin{aligned}(b, j)(a, i) &= (b, j)(1_\beta, j)(1_\alpha, i)(a, i)\\ &= (b, j)(1_{\beta\alpha}, ji)(a, i)\\ &= (b, j)(1_{\beta\alpha}, [a, ji]i)(a\varphi_{\alpha,\beta\alpha}, [a, ji]i)\\ &= (b\varphi_{\beta,\beta\alpha}, [a, ji]i)(a\varphi_{\alpha,\beta\alpha}, [a, ji]i)\\ &= (b\varphi_{\beta,\beta\alpha}a\varphi_{\alpha,\beta\alpha}, [a, ji]i)\\ &= (ba, [a, ji]i)\\ &= (b, j) \circ (a, i).\end{aligned}$$

This completed our proof.

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Fuzzy extension in BCI-algebras

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Abstract In this paper we recal the definition of non closed fuzzy algebraic structures namely fuzzy sub-halfalgebra of a BCI-algebra. Here we give the necessary and sufficient condition for the two fuzzy sub-halfalgebra to have open fuzzy extension.Also we recall yet another new notion of the concept called antifuzzy sub-halfalgebra and antifuzzy extension and obtain some characterization theorems about them.

Keywords Sub-halfalgebra, fuzzy extension, antifuzzy extension, open fuzzy extension.

§1. Introduction

The notion of BCK/BCI-algebras were interoduced by Iseki [3], and were extensively investigated by many researchers. They are two important classes of logical algebras.The notion of fuzzy sets was introduced by Zadeh [5] was applied to BCI-algwbras by Xi [4]. In this paper, we discuss the notion of fuzzy sub-halfalgebra and we also recall the notion of normalized fuzzy extension and maximal fuzzy extension and obtain a relation between these two concepts.

§2. Fuzzy extension

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$, satisfying the following conditions :

(I) $((x * y) * (x * z)) * (z * y) = 0$,

(II) $(x * (x * y)) * y = 0$,

(III) $x * x = 0$,

(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$,

for all $x, y, z, \in X$.

We can define a partial ordering " $\leq$ " on X by $x \leq y$ if and only if $x * y = 0$. Throught this paper X will always mean a BCI-algebra unless otherwise satisfied.

Definition 2.1. Let X be a BCI-algebra. A map $\mu : X \to [0, 1]$ is said to be a fuzzy sub-halfalgebra of X if $x * y \leq z$ implies $\mu(z) \geq min\{\mu(x), \mu(y)\}$ for all $x, y, z \in X$.

Example 2.2. Let $X = \left\{ \frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \cdots, \frac{1}{2^n} \right\}$, under usual Subtraction.

Define $\mu : X \to [0,1]$ by $\mu(x) = 1 - x$ for every $x \in X$. Now it is easy to verify that μ is a fuzzy sub-halfalgebra of X.

Theorem 2.3. Let μ be a fuzzy sub-halfalgebra of X and $x, y, z \in X$. Then $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ dose not in general imply $x * y \leq z$ in X.

Proof. In the above example we see $\mu(z) \geq \min\{\mu(x), \mu(y)\}$, but $y \neq z$ for $x = \frac{1}{2^0}$, $y = \frac{1}{2^1}$ and $z = \frac{1}{2^2}$.

Definition 2.4. Let μ and ν be any two fuzzy sub-halfalgebras of X such that

(i) $\mu(x) \leq \nu(x)$ for every $x \in X$.

(ii) if $x * y \leq z$ and $\mu(z) = \min\{\mu(x), \mu(y)\}$ then $\nu(z) = \min\{\nu(x), \nu(y)\}$. Then we say that ν is a fuzzy extension of μ.

Example 2.5. Let $X = \left\{\frac{1}{2^0}, \frac{1}{2^1}, \frac{1}{2^2}, \cdots, \frac{1}{2^n}\right\}$, be a BCI-algebra with respect to usual Subtraction.

Define $\mu : X \to [0,1]$ by $\mu(x) = 1 - x$ for every $x \in X$ and $\nu(x) = 1 - \frac{x}{2}$ for every $x \in X$ and it is easy to verify that ν is a fuzzy extension of μ.

Theorem 2.6. Let Let μ and ν be two fuzzy sub-halfalgebra of X. If ν is a fuzzy extension of μ and $x * y \leq z$ in X with $\nu(z) = \min\{\nu(x), \nu(y)\}$ then it need not in general imply $\mu(z) = \min\{\mu(x), \mu(y)\}$.

Proof. The proof is by an example. Take X and μ as in Example 2.5. Define $\nu : X \to [0,1]$ by $\nu(x) = 1$ for every $x \in X$. Now if we take $x = \frac{1}{2^0}$ and $y = \frac{1}{2^1}$ then $x * y = \frac{1}{2^0} - \frac{1}{2^1} = \frac{1}{2}$. It is easy to verify that $\nu(z) = 1$, $\min\{\nu(x), \nu(y)\} = 1$. $\mu(z) = \frac{1}{2}$ and $\min\{\mu(x), \mu(y)\} = 0$. This prove that $\nu(z) = \min\{\nu(x), \nu(y)\}$ and $\mu(z) \neq \min\{\mu(x), \mu(y)\}$. This completes the proof.

Definition 2.7. Let μ and ν be any two fuzzy sub-halfalgebra of X. Then the fuzzy extension ν (ofμ) is said to be an open fuzzy extention of μ if $x * y \leq z$ in X with $\mu(z) > \min\{\mu(x), \mu(y)\}$ implies $\nu(z) > \min\{\nu(x), \nu(y)\}$, ν is said to be closed fuzzy extension of μ if it is not an open fuzzy extension of μ.

Theorem 2.8. Every open fuzzy extension is a fuzzy extension but not conversely.

Proof. Every open fuzzy extension is a fuzzy extension, directly follows from the definitions of open fuzzy extension and fuzzy extension. Converse is not true.This is explicit by the following example.

Example 2.9. Let X be any BCI-algebra and μ be any fuzzy sub-halfalgebra of X. With $|Im(\mu)| \geq 2$. Then it is easy verify that $\nu : X \to [0,1]$ given by $\nu(x) = 1$ for every $x \in X$ is a fuzzy extension of μ. It is easily seen ν is not an open fuzzy extension of μ.

Definition 2.10. Let X be a BCI-algebra. A countable collection of fuzzy sub-halfalgebras of X, denoted by $\{\mu_i | i = 0, 1, 2, \cdots\}$ or $\{\mu_i\} \lim_{i=0}^{\infty}$ is called fuzzy extension chain for X if μ_{i+1} is a fuzy extension of μ_i for $i = 0, 1, 2, \cdots$. A fuzzy extension chain $\{\mu_i\} \lim_{i=0}^{\infty}$ is said to generate X if $\bigcup_{i=0}^{\infty} \mu_i = 1_X$ (Here 1_X denotes the map $\mu : X \to [0,1]$ such that $\mu(x) = 1$ for every $x \in X$). A fuzzy extension chain $\{\mu_i\} \lim_{i=0}^{\infty}$ is said to be a fuzzy extension chain for X if μ_{i+1} is an open fuzzy extension of μ_i for $i = 0, 1, 2, \cdots$.

Example 2.11. Let X be any BCI-algebra and $\mu_k : X \to [0,1]$ be defined by: $\mu_k(x) = \frac{k}{k+1}$ for every $k = 0, 1, 2, \cdots$. Then we get a countable collection of fuzzy sub-halfalgebra

$\{\mu_k\}\lim_{k=0}^{\infty}$ of X. If $x*y \leq z$ and $\mu_{k+1}(z) = \min\{\mu_{k+1}(x), \mu_{k+1}(y)\}$ then we have $\mu_k(z) = \min\{\mu_k(x), \mu_k(y)\}$. Hence μ_{k+1} is a fuzzy extension of μ_k for all $k = 0, 1, 2, \cdots$. This establishes that $\{\mu_k\}\lim_{k=0}^{\infty}$ is a fuzzy extension chain for X. Now we will prove $\{\mu_k\}\lim_{k=0}^{\infty}$ of X generates the X. Let x be an arbitrary element of the X then we have:

$$\left(\bigcup_{k=0}^{\infty}\mu_k\right)(x) = \sup_{k=0,1,2,\cdots}\{\mu_k(x)\} = \sup_{k=0,1,2,\cdots}\left\{\frac{k}{k+1}\right\} = 1 = 1_{X(x)}.$$

That is $(\bigcup\lim_{k=0}^{\infty}\mu_k)(x) = 1_{X(x)}$ for every $x \in X$. Thus we have proved that $(\bigcup\lim_{k=0}^{\infty}\mu_k)(x) = 1_X$. This implies that $\{\mu_k\}\lim_{k=0}^{\infty}$ generates the X.

Theorem 2.12. Let μ and ν be any two fuzzy sub-halfalgebra of X. If the fuzzy extension ν(of μ) is an fuzzy extension of μ then $\mu(z) = \min\{\mu(ux), \mu(y)\}$ if and only if $\nu(z) = \min\{\nu(x), \nu(y)\}$.

Proof. We prove the converse of the contra positive method. Let $x*y \leq z$ in X and $\mu(z) \neq \min\{\mu(x), \mu(y)\}$ to prove $\nu(z) \neq \min\{\nu(x), \nu(y)\}$, since μ is a fuzzy sub-halfalgebra of X, then we have $\mu(z) > \min\{\mu(x), \mu(y)\}$. Since ν is an open fuzzy extension of μ. This implies that $\nu(z) > \min\{\nu(x), \nu(y)\}$. That is $\nu(z) \neq \min\{\nu(x), \nu(y)\}$.

Hence we have proved that if $x*y \leq z$ in X and $\mu(z) \neq \min\{\mu(x), \mu(y)\}$ then $\nu(z) \neq \min\{\nu(x), \nu(y)\}$. This proves that if the fuzzy extension ν (μ) is an open fuzzy extension μ then $\mu(z) = \min\{\mu(x), \mu(y)\}$ if and only if $\nu(z) = \min\{\nu(x), \nu(y)\}$ for every $x*y \leq z$ in X.

§3. Antifuzzy extension

Now we proceed on to give the definition of antifuzzy extension.

Definition 3.1 A fuzzy subset μ of X is said to be antifuzzy sub-halfalgebra of X if $x*y \leq z$ in X implies $\mu(z) \leq \max\{\mu(x), \mu(y)\}$.

Definition 3.2. Let μ and ν be two antifuzzy sub-halfalgebras of X. Then ν is said to be an antifuzzy extension of μ if the following two condition hold:

(i)$\mu(x) \geq \nu(x)$ for every $x \in X$,

(ii) if $x*y \leq z$ and $\mu(z) = \max\{\mu(x), \mu(y)\}$ then $\nu(z) = \max\{\nu(x), \nu(y)\}$. As in case of fuzzy extension we can prove that if ν is an antifuzzy extension of μ and $x*y \leq z$ and $\nu(z) = \max\{\nu(x), \nu(y)\}$ then it need not in general imply that $\mu(z) = \max\{\mu(x), \mu(y)\}$.

Definition 3.3. Let μ and ν be two antifuzzy sub-halfalgebras of X. Then the antifuzzy extention ν (of μ) is said to be an open antifuzzy extension of μ if: $x*y \leq z$ and $\mu(z) < \max\{\mu(x), \mu(y)\}$ then $\nu(z) < \max\{\nu(x), \nu(y)\}$. If ν is not an open antifuzzy extension of μ then we say that ν is a closed antifuzzy extension of μ.

Definition 3.4. Let X be a BCI-algebra. A countable collection of antifuzzy sub-halfalgebra of X , denoted by denoted by $\{\mu_i | i = 0, 1, 2, \cdots\}$ or $\{\mu_i\}\lim_{i=0}^{\infty}$ is called an antifuzzy extension chain for X if μ_{i+1} is an antifuzzy extension of μ_i for $i = 0, 1, 2, \cdots$.

Example 3.5. Let $P = \left\{\frac{1}{2^0}, \frac{1}{2^1}, \cdots, \frac{1}{2^n}\right\}$ take usual subtraction in X. Define $\mu : X \to [0, 1]$ by $\mu(x) = x$ for every $x \in X$. It is clear that μ is an antifuzzy sub-halfalgebra of X.

Define μ , $\nu : X \to [0, 1]$ by: $\mu(x) = x$ for all $x \in X$ $\nu(x) = \frac{x}{2}$ for all $x \in X$. Then it is easy to verify that ν is an antifuzzy extension of μ.

Theorem 3.6 Let X be a BCI-algebra. Then μ is a fuzzy sub-halfalgebra of X if and only if μ^c is an antifuzzy sub-halfalgebra of X.

Proof. Let μ be a fuzzy sub-halfalgebra of X and $x * y \leq z$ in X. Since $\mu^c(x) = 1 - \mu(x)$ for all $x \in X$, we nave $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ if and only if $\mu^c(z) \leq \max\{\mu^c(x), \mu^c(y)\}$. This prove that μ is a fuzzy sub-halfalgebra of X if and only if μ^c is an antifuzzy sub-halfalgebra of X.

Theorem 3.7. Let μ and ν be two fuzzy sub-halfalgebra of X. Then:

(i) ν is a fuzzy extension of μ if and only if ν^c is an antifuzzy extention of μ^c.

(ii) ν an open fuzzy extension of μ if and only if ν^c is an open antifuzzy extension of μ^c.

Proof. Let ν be a fuzzy extension of μ to prove ν^c is an antifuzzy extension of μ^c. Since ν is a fuzzy extension of μ, we have $\mu(x) \leq \nu(x)$ for all $x \in X$ that is $\mu^c(x) \geq \nu^c(x)$ for all $x \in X$. Let $x*y \leq z$ in X and $\mu^c(z) = \max\{\mu^c(x), \mu^c(y)\}$ that is $1-\mu(z) = \max\{1-\mu(x), 1-\mu(y)\}$. Then $1 - \mu(z) = 1 - \min\{\mu(x), \mu(y)\}$. Hence $\mu(z) = \min\{\mu(x), \mu(y)\}$. Since ν is a fuzzy extension of μ and $x * y \leq z$ in X, we have $\nu(z) = \min\{\nu(x), \nu(y)\}$. Using the properties of Min and Max function we have $1-\nu(z) = \max\{1-\nu(x), 1-\nu(y)\}$. That is $\nu^c(z) = \max\{\nu^c(x), \nu^c(y)\}$. Thus ν^c is an antifuzzy extension of μ^c.

(ii) Let ν be an open fuzzy extension of μ. To prove ν^c is an open antifuzzy extension of μ^c. Since ν is an open fuzzy extension of μ we have $\mu(x) \leq \nu(x)$ for all $x \in X$ that is $\mu^c(x) \geq \nu^c(x)$ for all $x \in X$. Let $x * y \leq z$ in X and $\mu^c(z) < \max\{\mu^c(x), \mu^c(y)\}$ then $1 - \mu(z) < \max\{1-\mu(x), 1-\mu(y)\}$. This implies $1 - \mu(z) < 1 - \min\{\mu(x), \mu(y)\}$. That is $\mu(z) > \min\{\mu(x), \mu(y)\}$. As ν is an open fuzzy extension of μ and $x * y \leq z$. We have $\nu(z) > \min\{\nu(x), \nu(y)\}$ then $1 - \mu(z) < \max\{1-\mu(x), 1-\mu(y)\}$ that is $\nu^c(z) < \max\{\nu^c(x), \nu^c(y)\}$. Thus ν^c is an open antifuzzy extention of μ^c. Conversely let ν^c be an open extension of μ^c then we prove that ν is an open fuzzy extension of μ using the fact $1-\max\{a, b\} = \min\{1-a, 1-b\}$ for all $a, b \in [0, 1]$.

Example 3.8. Choose μ and ν the example3.5 where $P = \left\{\frac{1}{2^0}, \frac{1}{2^1}, \cdots, \frac{1}{2^n}\right\}$, $\mu(x) = 1-x$ and $\nu(x) = 1 - \frac{x}{2}$ for all $x \in X$. It is easy to verify that ν is a fuzzy extension of μ. Now we calculate μ^c and ν^c as $\mu^c(x) = x$ and $\nu(x) = \frac{x}{2}$, for every $x \in X$. Clearly ν^c is an antifuzzy extension of μ^c. This shows that ν is a fuzzy extention of μ if and only if ν^c is an antifuzzy extension of μ^c.

§4. Maximal fuzzy extension

Now we proceed on to define the concept of maximal fuzzy extension.

Definition 4.1 Let μ and ν be any two fuzzy sub-halfalgebra of X. A fuzzy extension $\nu(of\mu)$ is said to be maximal fuzzy extention of μ if there dos not exist any fuzzy sub-halfalgebra f of X such that $\nu \subset f$.

Example 4.2 Let $P = \{1, 2, 3, \cdots, n\}$ be a BCI-algebra with respect to usual subtraction. Define $\mu(x) = \frac{1}{2}$ and $\nu(x) = 1$ for all $x \in X$. It is easy to verify that both μ and ν are fuzzy sub-halfalgebra of X. Now we cannot find any fuzzy sub-halfalgebra f of X such that $\nu \subset f$. Hence ν is the maximal fuzzy extension of μ.

Definition 4.3 Let μ and ν be any two fuzzy sub-halfalgebra of X. A fuzzy extension $\nu(of\mu)$ is said to be a normalized fuzzy extension of μ if $\nu(x) = 1$ for some $x \in X$. In the above example the fuzzy sub-halfalgebrs ν is a normalized fuzzy extension of μ.

Definition 4.4. Let μ be a fuzzy subset of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. A map $\mu_\alpha^T(x) : X \to [0, 1]$ is called a fuzzy translation of μ if $\mu_\alpha^T(x) = \mu(x) + \alpha$ for all $x \in X$.

Example 4.5. Let $X = \{2, 4, \cdots, 2n\}$. Define $\mu : X \to [0, 1]$ by $\mu(x) = \dfrac{1}{x}$ for all $x \in X$. Then for $\alpha = \dfrac{1}{4}$ we have $\mu_\alpha^T(x) = \dfrac{1}{x} + \dfrac{1}{4}$ for all $x \in X$. It is easy to verify that μ_α^T is a fuzzy translation of μ.

Theorem 4.6. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$ then every fuzzy translation μ_α^T of μ is a fuzzy sub-halfalgebra of X.

Proof. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. For $x * y \leq z$ in X , we have $\mu(z) \geq \min\{\mu(x), \mu(uy)\}$ that is $\alpha + \mu(z) \geq \alpha + \min\{\mu(x), \mu(y)\}$ that is $\alpha + \mu(z) \geq \min\{\alpha + \mu(x), \alpha + \mu(y)\}$ by the definition of fuzzy translation $\mu_\alpha^T(z) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$. That is μ_α^T is a fuzzy sub-halphalgebra of X.

Theorem 4.7. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. Then every fuzzy translation $\mu_\alpha^T(of\mu)$ is a fuzzy extension of μ.

Proof. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$, we know by Theorem4.6 μ_α^T is a fuzzy sub-halfalgebra of X. Clearly $\mu_\alpha^T(x) \geq \mu(x)$ for all $x \in X$. Further if $x * y \leq z$ in X , and $\mu(uz) = \min\{\mu(x), \mu(y)\}$ then $\mu_\alpha^T(z) = \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$. Hence μ_α^T is a fuzzy extension of μ.

Example 4.8. Let $X = \left\{\dfrac{1}{3^0}, \dfrac{1}{3^1}, \cdots, \dfrac{1}{3^n}\right\}$ be a BCI-algebra with respect to usual subtraction. Define $\mu : X \to [0, 1]$ by $\mu(x) = 1 - x$ for all $x \in X$. Now it can be easily checked that μ is a fuzzy sub-halfalgebra of X. For $\alpha = \dfrac{1}{3}$ we have $\mu_\alpha^T(x) = 1 - x + \dfrac{1}{3}$ for all $x \in X$. It is easy to verify that μ_α^T is a fuzzy extension of μ.

Theorem 4.9. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. If μ_α^T is fuzzy translation of μ then:

(i) μ_α^T is a maximal fuzzy extension of μu if and only if μ is a constant map.

(ii) If μ has sup property then μ_α^T is normalized fuzzy extension of μ but not conversly.

Proof. (i) Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - sup\{\mu(x) : x \in X\}]$. Then by Theorem 4.7 μ_α^T is a fuzzy extension of μ. Let $\mu^u T_\alpha$ be a maximal fuzzy extension of μ then $\mu_\alpha^T(x) = 1$ for all $x \in X$, that is $\mu(uux) = \sup\{\mu(y)|y \in X\}$ for every $x \in X$. Hence μ is a constant map. Conversly let μ be a constant map, that is $\mu(x) = \beta$ for all $x \in X$ (where β is a fixed element of $[0, 1]$).

Now consider the fuzzy translation μ_α^T of μ, $\mu_\alpha^T(x) = \mu(ux) + \alpha = \beta + 1 - \beta = 1$ for all $x \in X$. Hence μ_α^T is a maximal fuzzy extension of μ. (ii) Let μ have sup property, that is for every subset T of X there exists $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t)|t \in T\}$. If possible let as assume that μ_α^T is not a normalized fuzzy extension of μ then $\mu_\alpha^T(x) < 1$ for all $x \in X$. That is $\mu(x) < \sup\{\mu(y)|y \in X\}$ for all $x \in X$. Since μ has supproperty for every subset T of X there exists $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t)|t \in T\}$. So we get $\mu(x) < \mu(t_0)$ which is a contradiction.

Hence μ_α^T is a normalized fuzzy extension of μ. If μ_α^T is a normalized fuzzy extension of μ

then it does not in general imply that μ has sup property. This can be seen from the example given below.

Consider $X = [0,1]$ under usual subtraction. Define $\mu : X \to [0,1]$ by $\mu(x) = 1 - x$ for all $x \in X$. It is easy to see that μ is a fuzzy sub-halfalgebra of X. Let $\alpha \in [0, 1 - sup\{\mu(x) : x \in X\}]$ then μ_α^T is a normalized fuzzy extension of μ for there exists $0 \in X$ such that $\mu_\alpha^T(0) = 1$. Take $T = [0,1]$ then $T \subseteq [0,1]$ and $\sup\{\mu(t)|t \in T\} = 1$. Now it is easy to verify that there is no $t_0 \in T$ such that $\mu(t_0) = 1$. Hence μ dosenot have sup property.

Definition 4.10. A fuzzy subset μ of a set X has the weak sup property if there exist $x_0 \in X$ such that $\mu(x_0) = \sup\{\mu(x)|x \in X\}$.

Theorem 4.11 Let μ be a fuzzy subset of X. If μ has sup property then μ has weak sup property but not conversly.

Proof. Let μ be a fuzzy set of X. If μ has sup property then by the definition of sup property for every subset T of X there exist $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(t)|t \in T\}$. The above condition is true for every subset T of X. If we take T as X then there exists $t_0 \in T$ such that $\mu(t_0) = \sup\{\mu(x)|x \in X\}$.

Hence μ has weak sup property.However the converse is not true this can be seen by the following example. Consider the set $[\alpha, \beta]$ where α and β are any two arbitary fixed numberes in interval $[0,1]$ with $\alpha < \beta$. Define $\mu : [\alpha, \beta] \to [0,1]$ by $\mu(x) = x$ for all $x \in X$.

Then $\mu(\beta) = \sup\{\mu(x)|x \in X\}$ but for $T = [\alpha, \beta)$ we have $\sup\{\mu(x)|x \in T\} = \beta$ and there is no $t_0 \in T$ such that $\mu(t_0) = \beta$. That is μ has weak sup property but μ dose not have sup property.

Theorem 4.12. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. Then μ_α^T is a normalized fuzzy extension of μ if and only if μ has weak supproperty.

Proof. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. Let μ_α^T be a normalized fuzzy extension of μ hen by he definition of normalized fuzzy extension we have $\mu_\alpha^T(x_0) = 1$ for some $x_0 \in X$. That is $\mu(x_0) = \sup\{\mu(x)|x \in X\}$ for some $x_0 \in X$. Hence μ has weak sup property. Conversly let μ have weak supproperty.

Then there exists $x_0 \in X$ such that $\mu(x_0) = \sup\{\mu(x)|x \in X\}$. Now consider the fuzzy translation μ_α^T of μ. $\mu_\alpha^T(x) = \mu(x) + 1 - \sup\{\mu(x)|x \in X\}$. Take $x = x_0$, then we have $\mu_\alpha^T(x_0) = \mu(x_0) + 1 - \mu(x_0) = 1$. Hence μ_α^T is normalized fuzzy extension of μ. This completes the proof of the theorem.

Definition 4.13. Let μ be a fuzzy subset of X and $\beta \in [0,1]$. A map $\mu_\beta : X \to [0,1]$ is called a fuzzy multiplication of μ if $\mu_\beta(x) = \beta.\mu(x)$ for all $x \in X$.

Theorem 4.14. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$ then every fuzzy translation $\mu_\alpha^T(of\mu)$ is a fuzzy extension of fuzzy multiplication $\mu_\beta(of\mu)$.

Proof. Let μ be a fuzzy sub-halfalgebra of X. If $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$ then fuzzy translation μ_α^T and the fuzzy multiplication μ_β are fuzzy sub-halfalgebras of X. We see that for all $x \in X$, $\mu_\beta = \beta.\mu(x) \leq \mu(x) \leq \alpha + \mu(x) = \mu_\alpha^T(x)$.
Also if $x*y \leq z$ and $\mu_\beta(z) = \min\{\mu_\beta(x), \mu_\beta(y)\}$ then this implies $\beta.\mu(z) = \beta.\min\{\mu_\beta(x), \mu_\beta(y)\}$ that is $\mu(z) = \min\{\mu(x), \mu(y)\}$ that is $\alpha + \mu(z) = \min\{\alpha + \mu(x), \alpha + \mu(y)\}$ that is $\mu_\alpha^T(z) = \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$. Hence μ_α^T is a fuzzy extension of μ_β for all $\alpha \in [0, 1 - \sup\{\mu(x) : x \in X\}]$. Hence the theorem. It is to be noted that for $\beta = 0$ the result is not true.

Theorem 4.15. Let μ be a fuzzy sub-halfalgebra of X and $\alpha \in [0, 1-\sup\{\mu(x) : x \in X\}]$. hen $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ where ($S = \{\gamma : 0 \leq \gamma \leq \alpha\}$, and γ rational) is an open fuzzy extension chain for X. Further $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ generates the BCI-subhalf algebra X if and only if μ is a constant map.

Proof. It is easy to see from the proof of the Theorem 4.14 $k \in S$ and μ_k^T is a fuzzy sub-halfalgebra of X. Choose $k_1,\ k_2 \in S$ such that $0 \leq k_1 \leq k_2 \leq k$ then $\mu_{k_1}^T(x) \leq \mu_{k_2}^T(x)$ for all $x \in X$. Thus for $k_i, k_j \in S(i \leq j)$, $\mu_{k_i}^T(x) \leq \mu_{k_j}^T(x)$ for all $x \in X$ without loos of generality we have $0 \leq k_0 \leq k_1 \leq \cdots \leq k_i \leq k_{i+1} \cdots \leq k$.

Then by the above construction $\mu_{k_0}^T(x) \leq \mu_{k_j}^T(x) \leq \cdots \leq \mu_{k_i}^T(x) \leq \mu_{k_{i+1}}^T(x) \leq \cdots \leq \mu_k^T(x)$ for all $x \in X$. Now if $x * y \leq z$ and $\mu_{k_i}^T(z) = \min\{\mu_{k_i}^T(x), \mu_{k_i}^T(y)\}$, then it is easy to see that $\mu_{k_{i+1}}^T(z) = \min\left\{\mu_{k_{i+1}}^T(x), \mu_{k_{i+1}}^T(y)\right\}$ for $i = 0, 1, 2, \cdots$. Further if $x * y \leq z$ and $\mu_{k_i}^T(z) > \min\{\mu_{k_i}^T(x), \mu_{k_i}^T(y)\}$, then we can $\mu_{k+1}^T(z) > \min\{\mu_{k+1}^T(x), \mu_{k+1}^T(y)\}$for $i = 0, 1, 2, \cdots$. Hence $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ is an open fuzzy extension chain for X.

Now we will prove that $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ generate the BCI-algebra X if and only if μ is a constant map. Let $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ generate the BCI-algebra X. Then we have $\bigcup\lim_{k\in S}^{\infty}\mu_k^T = 1_X$ that is $\sup\{\mu_k^T(x)|k \in S\} = 1$ for every $x \in X$, that is $\sup\{\mu(x) + k|k \in S\} = 1$ for all $x \in X$, that is $\mu(x) + 1 - \sup\{\mu(y)|y \in X\} = 1$ for all $x \in X$. That is $\mu(x) = \sup\{\mu(y)|y \in X\}$ for all $x \in X$. This proves that μ is a constant map.

Conversely let μ be a constant map that is $\mu(x) = \gamma$ for all $x \in X$(where γ is fixed elements in $[0, 1]$). Consider $(\bigcup\lim_{k\in S}^{\infty}\mu_k^T)(x)$ for an arbitrary $x \in X$.

$$\begin{aligned}(\bigcup\lim_{k\in S}^{\infty}\mu_k^T)(x) &= \sup\{\mu_k^T(x)|k \in S\} = \sup\{\mu(x) + k|k \in S\} \\ &= \mu(x) + 1 - sup\{\mu(y)|y \in X\} = \gamma + 1 - \gamma = 1 = 1_{X(x)}\end{aligned}$$

since x is an arbitary element of X we have $(\bigcup\lim_{k\in S}^{\infty}\mu_k^T)(x) = 1_{X(x)}$ for all $x \in X$. Hence $\{\mu_k^T\}\lim_{k\in S}^{\infty}$ generates the BCI-algebra X.

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The Pell's equation $x^2 - Dy^2 = \pm a$

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Abstract Let $D \neq 1$ be a positive non-square integer. Let a be a positive integer. In this paper is considered the Pell's equation $x^2 - Dy^2 = \pm a$ and some recurrence relations are obtained.

Keywords Pell's equation, solutions of Pell's equation.

§1. Introduction

Let $D \neq 1$ be any positive non-square integer and a be any fixed integer. The equation

$$x^2 - Dy^2 = \pm a$$

is known as Pell's equation. It is named mistakenly after John Pell (1611-1685) who was a mathematician who in fact did not contribute for solving it (see [8]).

For $a = 1$, the Pell's equation $x^2 - Dy^2 = \pm 1$ is known as classical Pell's equation and it has infinitely many solutions (x_n, y_n) for $n \geq 1$. There are different methods for finding the first non-trivial (x_1, y_1) solution called the fundamental solution from which all others are easily computed (see [3], and [9]).

There are many papers in which different types of Pell's equation are considered (see [1], [2], [4], [5], [6], [7]).

In his paper [1] A. Tekcan considered the equation $x^2 - Dy^2 = \pm 4$ and obtained some formulas for that equation.

In this paper we will consider a more general equation, indeed $x^2 - Dy^2 = \pm a$, with a positive integer. Some similar results to those in [1] and [2] will be obtained. The ideas used in [1] and [2] are going to be used again in order to prove the main results.

§2. The pell's equation $x^2 - Dy^2 = a$

Theorem 1.1. Let (x_1, y_1) be the fundamental solution of the Pell's equation $x^2 - Dy^2 = a$ and let

$$\begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1}$$

for $n \geq 1$. Then the integer solutions of the Pell's equation $x^2 - Dy^2 = \pm a$ are given by

$$(x_n, y_n) = \left(\frac{u_n}{\sqrt{a}^{n-1}}, \frac{v_n}{\sqrt{a}^{n-1}} \right). \tag{2}$$

Proof. We prove the theorem using the method of mathematical induction on n.

Clearly it is true for $n = 1$. We assume that $x^2 - Dy^2 = a$ is satisfied for (x_{n-1}, y_{n-1}), i.e.

$$x_{n-1}^2 - Dy_{n-1}^2 = \frac{u_{n-1}^2}{(\sqrt{a}^{n-2})^2} - \frac{Dv_{n-1}^2}{(\sqrt{a}^{n-2})^2} = \frac{u_{n-1}^2 - Dv_{n-1}^2}{a^{n-2}} = a. \tag{3}$$

From (3) we have x and y should have index n (See the original) $u_{n-1}^2 Dv_{n-1}^2 = a^{n-1}$.

Then we show that $x_n^2 - Dy_n^2 = \pm a$.

First we note that

$$\begin{aligned}
\begin{pmatrix} u_n \\ v_n \end{pmatrix} &= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^n \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix} \cdot \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{n-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix} \begin{pmatrix} u_{n-1} \\ v_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} x_1 u_{n-1} + Dy_1 v_{n-1} \\ y_1 u_{n-1} + x_1 v_{n-1} \end{pmatrix}.
\end{aligned}$$

So

$$\begin{cases} u_n = x_1 u_{n-1} + Dy_1 v_{n-1} \\ v_n = y_1 u_{n-1} + x_1 v_{n-1} \end{cases} \tag{4}$$

Then

$$\begin{aligned}
& x_n^2 - Dy_n^2 \\
= \; & \frac{u_n^2 - Dy_n^2}{(\sqrt{a}^{n-1})^2} \\
= \; & \frac{(x_1 u_{n-1} + Dy_1 v_{n-1})^2 - D(y_1 u_{n-1} + x_1 v_{n-1})^2}{a^{n-1}} \\
= \; & \frac{x_1^2 u_{n-1}^2 + 2x_1 u_{n-1} Dy_1 v_{n-1} + D^2 y_1^2 v_{n-1}^2 - D(y_1^2 u_{n-1}^2 + 2y_1 u_{n-1} x_1 v_{n-1} + x_1^2 v_{n-1}^2)}{a^{n-1}} \\
= \; & \frac{(x_1^2 - Dy_1^2) u_{n-1}^2 - Dv_{n-1}^2 (x_1^2 - Dy_1^2)}{a^{n-1}} \\
= \; & \frac{(x_1^2 - Dy_1^2)(u_{n-1}^2 - Dv_1^2)}{a^{n-1}} \\
= \; & \frac{a \cdot a^{n-1}}{a^{n-1}} \\
= \; & a
\end{aligned}$$

completing the proof.

From the theorem 1.1, the following corollary can be proved.

Corollary 1.2. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = a$, then

$$x_n = \frac{x_1 x_{n-1} + D y_1 y_{n-1}}{\sqrt{a}}; \quad y_n = \frac{y_1 x_{n-1} + x_1 y_{n-1}}{\sqrt{a}} \tag{5}$$

and

$$\begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} = -\sqrt{a} y_1,$$

for $n \geq 2$.

Proof. By (4) $u_n = x_1 u_{n-1} + D y_1 v_{n-1}$; $v_n = y_1 u_{n-1} + x_1 v_{n-1}$.

By (2) $u_n = \sqrt{a}^{n-1} \cdot x_n$; $v_n = \sqrt{a}^{n-1} \cdot y_n$, so then $u_{n-1} = \sqrt{a}^{n-2} \cdot x_{n-1}$, $v_{n-1} = \sqrt{a}^{n-2} \cdot y_{n-1}$.

Hence we get:

$$\begin{aligned} \sqrt{a}^{n-1} \cdot x_n &= x_1 \cdot \sqrt{a}^{n-2} \cdot x_{n-1} + D y_1 \sqrt{a}^{n-2} \cdot y_{n-1} \\ &= \sqrt{a}^{n-2}(x_1 x_{n-1} y_n + D y_1 y_{n-1}). \end{aligned}$$

It follows that $x_n = \dfrac{x_1 x_{n-1} + D y_1 y_{n-1}}{\sqrt{a}}$.

In a similar way, one obtains:

$$\begin{aligned} \sqrt{a}^{n-1} y_n &= y_1 \sqrt{a}^{n-2} \cdot x_{n-1} + x_1 \sqrt{a}^{n-2} \cdot y_{n-1} \\ &= \sqrt{a}^{n-2}(y_1 x_{n-1} + x_1 y_{n-1}). \end{aligned}$$

It follows that $y_n = \dfrac{y_1 x_{n-1} + x_1 y_{n-1}}{\sqrt{a}}$.

Finally,

$$\begin{aligned} \begin{vmatrix} x_n & x_{n-1} \\ y_n & y_{n-1} \end{vmatrix} &= x_n y_{n-1} - y_n x_{n-1} \\ &= \frac{x_1 x_{n-1} + D y_1 y_{n-1}}{\sqrt{a}} y_{n-1} - \frac{y_1 x_{n-1} + x_1 y_{n-1}}{\sqrt{a}} x_{n-1} \\ &= \frac{-y_1(x_{n-1}^2 - D y_{n-1}^2)}{\sqrt{a}} \\ &= -\frac{a y_1}{\sqrt{a}} = -\sqrt{a} y_1, \end{aligned}$$

completing the proof.

Remark. For $a = 4$, (2) and (5) represent the results of [1].

Now following the same argument of proof as in [2] we can prove the following theorem:

Theorem 1.3. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = a$ then (x_n, y_n) satisfy the following recurrence relations

$$\begin{cases} x_n = (x_1 - 1)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n = (x_1 - 1)(y_{n-1} + y_{n-2}) - y_{n-3} \end{cases} \tag{6}$$

for $n \geq 4$.

Proof. It is a matter of tedious calculations to show that relations (6) are true for $n = 4$.(See [2])

Next, we assume that (6) holds for n and we show that it holds for $n+1$.

Indeed, by (5) and by hypothesis we have:

$$\begin{aligned} x_{n+1} &= \frac{x_1((x_1-1)(x_{n-1}+x_{n-2})-x_{n-3})+Dy_1((x_1-1)(y_{n-1}+y_{n-2})-y_{n-3})}{\sqrt{a}} \\ &= (x_1-1)\left(\frac{x_1(x_{n-1}+x_{n-2})+Dy_1(y_{n-1}+y_{n-2})}{\sqrt{a}}\right)-\frac{x_1x_{n-3}+Dy_1y_{n-3}}{\sqrt{a}} \\ &= (x_1-1)\left(\frac{x_1x_{n-1}+Dy_1y_{n-1}}{\sqrt{a}}+\frac{x_1x_{n-2}+Dy_1y_{n-2}}{\sqrt{a}}\right)-x_{n-2} \\ &= (x_1-1)(x_n+x_{n-1})-x_{n-2}, \end{aligned}$$

$$\begin{aligned} y_{n+1} &= \frac{y_1((x_1-1)(x_{n-1}+x_{n-2})-x_{n-3})+x_1((x_1-1)(y_{n-1}+y_{n-2})-y_{n-3})}{\sqrt{a}} \\ &= (x_1-1)\left(\frac{y_1(x_{n-1}+x_{n-2})+x_1(y_{n-1}+y_{n-2})}{\sqrt{a}}\right)-\frac{y_1x_{n-3}+x_1y_{n-3}}{\sqrt{a}} \\ &= (x_1-1)\left(\frac{y_1x_{n-1}+x_1y_{n-1}}{\sqrt{a}}+\frac{y_1x_{n-2}+x_1y_{n-2}}{\sqrt{a}}\right)-y_{n-2} \\ &= (x_1-1)(y_n+y_{n-1})-y_{n-2}, \end{aligned}$$

completing the proof.

§3. The negative Pell's equation $x^2 - Dy^2 = -a$

Theorem 2.1. If (x_1, y_1) is the fundamental solutions of the negative Pell's equation $x^2 - Dy^2 = -a$, then the other solution are (x_{2n+1}, y_{2n+1}) where

$$(x_{2n+1}, y_{2n+1}) = \left(\frac{u_{2n+1}}{a^n}, \frac{v_{2n+1}}{a^n}\right), \tag{7}$$

for $n \geq 0$.

Proof. Clearly it is true for $n = 0$.

We assume that $x^2 - Dy^2 = -a$ holds for (x_{2n-1}, y_{2n-1}) so

$$x_{2n-1}^2 - Dy_{2n-1}^2 = \frac{u_{2n-1}^2 - Dv_{2n-1}^2}{a^{2n-2}} = -a. \tag{8}$$

From (8), we have $u_{2n-1}^2 - Dv_{2n-1}^2 = -a^{2n-1}$.

Then we show that $x_{2n+1}^2 - Dy_{2n+1}^2 = -a$.

First we note that

$$\begin{aligned}
\begin{pmatrix} u_{2n+1} \\ v_{2n+1} \end{pmatrix} &= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n+1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \cdot \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2n-1} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} x_1 & Dy_1 \\ y_1 & x_1 \end{pmatrix}^{2} \begin{pmatrix} u_{2n-1} \\ v_{2n-1} \end{pmatrix} \\
&= \begin{pmatrix} u_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1v_{2n-1} \\ 2x_1y_1u_{2n-1} + (x_1^2 + Dy_1^2)v_{2n-1} \end{pmatrix}.
\end{aligned}$$

So

$$\begin{cases} u_{2n+1} = u_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1v_{2n-1} \\ v_{2n+1} = 2x_1y_1u_{2n-1} + (x_1^2 + Dy_1^2)v_{2n-1} \end{cases} \tag{9}$$

Using (9) we obtain:

$$\begin{aligned}
& x_{2n+1}^2 - Dy_{2n+1}^2 \\
= \ & \frac{u_{2n+1}^2 - Dv_{2n+1}^2}{a^{2n}} \\
= \ & \frac{(u_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1v_{2n-1})^2 - D(2x_1y_1u_{2n-1} + (x_1^2 + Dy_1^2)v_{2n-1})^2}{a^{2n}} \\
= \ & \frac{u_{2n-1}^2(x_1^2 + Dy_1^2)^2 - Dv_{2n-1}^2(x_1^2 + Dy_1^2)^2 + 4D^2x_1^2y_1^2v_{2n-1}^2 - 4Du_{2n-1}^2x_1^2y_1^2}{a^{2n}} \\
= \ & \frac{(x_1^2 + Dy_1^2)^2(u_{2n-1}^2 - Dv_{2n-1}^2) - 4Dx_1^2y_1^2(u_{2n-1}^2 - Dv_{2n-1}^2)}{a^{2n}} \\
= \ & \frac{(u_{2n-1}^2 - Dv_{2n-1}^2)(x_1^2 - Dy_1^2)^2}{a^{2n}} \\
= \ & \frac{-a^{2n-1} \cdot a^2}{a^{2n}} \\
= \ & -a.
\end{aligned}$$

From the theorem 2.1, the following corollary can be proved.

Corollary 2.2. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = -a$ then:

$$\begin{aligned}
x_{2n+1} &= \frac{(x_1^2 + Dy_1^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{a}, \\
y_{2n+1} &= \frac{2x_1y_1x_{2n-1} + (x_1^2 + Dy_1^2)y_{2n-1}}{a}
\end{aligned} \tag{10}$$

and

$$\begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} = 2x_1y_1$$

for $n \geq 1$.

Proof. By (9)

$$u_{2n+1} = u_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1v_{2n-1},$$
$$v_{2n+1} = 2x_1y_1u_{2n-1} + (x_1^2 + Dy_1^2)v_{2n-1}.$$

By (7)

$$u_{2n+1} = a^n x_{2n+1}, \quad v_{2n+1} = a^n y_{2n+1},$$

so then

$$u_{2n-1} = a^{n-1}x_{2n-1}, \quad v_{2n-1} = a^{n-1}y_{2n-1}.$$

Hence we get

$$\begin{aligned} a^n x_{2n+1} &= a^{n-1}x_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1a^{n-1}y_{2n-1} \\ &= a^{n-1}(x_{2n-1}(x_1^2 + Dy_1^2) + 2Dx_1y_1y_{2n-1}). \end{aligned}$$

It follows that

$$x_{2n+1} = \frac{(x_1^2 + Dy_1^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{a}.$$

In a similar way we obtain

$$\begin{aligned} a^n y_{2n+1} &= 2a^{n-1}x_{2n-1}x_1y_1 + (x_1^2 + Dy_1^2)a^{n-1}y_{2n-1} \\ &= a^{n-1}(2x_{2n-1}x_1y_1 + (x_1^2 + Dy_1^2)y_{2n-1}). \end{aligned}$$

It follows that

$$y_{2n+1} = \frac{2x_1y_1x_{2n-1} + (x_1^2 + Dy_1^2)y_{2n-1}}{a}.$$

Finally,

$$\begin{aligned} & \begin{vmatrix} x_{2n+1} & x_{2n-1} \\ y_{2n+1} & y_{2n-1} \end{vmatrix} \\ = \; & x_{2n+1}y_{2n-1} - y_{2n+1}x_{2n-1} \\ = \; & \frac{(x_1^2 + Dy_1^2)^2x_{2n-1} + 2Dx_1y_1y_{2n-1}}{a}y_{2n-1} - \frac{2x_1y_1x_{2n-1} + (x_1^2 + Dy_1^2)y_{2n-1}}{a}x_{2n-1} \\ = \; & \frac{2Dx_1y_1y_{2n-1}^2 - 2x_1y_1x_{2n-1}^2}{a} \\ = \; & \frac{-2x_1y_1(x_{2n-1}^2 - Dy_{2n-1}^2)}{a} \\ = \; & 2x_1y_1, \end{aligned}$$

completing the proof.

Remark. (7) and (10) represent a general case of the results of [1].

Now following the same argument of proof as in [2] we can prove the following theorem:

Theorem 1.6. If (x_1, y_1) is the fundamental solution of the Pell's equation $x^2 - Dy^2 = -a$, then (x_{2n+1}, y_{2n+1}) satisfy the following recurrence relations

$$\begin{cases} x_{2n+1} = (x_1^2 + 1)(x_{2n-1} + x_{2n-3}) - x_{2n-5} \\ y_{2n+1} = (x_1^2 + 1)(y_{2n-1} + y_{2n-3}) - y_{2n-5} \end{cases} \tag{11}$$

for $n \geq 3$.

Proof. Tedious calculations, yield that relations (11) are true for $n = 3$,(see [2]).

Now we assume that (x_{2n+1}, y_{2n+1}) satisfies(11) and we prove that also (x_{2n+3}, y_{2n+3}) satisfies(11).

Indeed by (10) and hypothesis we obtain.

$$\begin{aligned}
& x_{2n+3} \\
= & \frac{(x_1^2 + Dy_1^2)((x_1^2+1)(x_{2n-1} + x_{2n-3}) - x_{2n-5})}{a} \\
+ & \frac{2Dx_1y_1((x_1^2+1)(y_{2n-1} + y_{2n-3}) - y_{2n-5})}{a} \\
= & (x_1^2+1)\frac{(x_1^2 + Dy_1^2)(x_{2n-1} + x_{2n-3}) + 2Dx_1y_1(y_{2n-1} + y_{2n-3})}{a} \\
- & \frac{(x_1^2 + Dy_1^2)x_{2n-5} + 2Dx_1y_1y_{2n-5}}{a} \\
= & (x_1^2+1)\left(\frac{(x_1^2 + Dy_1^2)x_{2n-1} + 2Dx_1y_1y_{2n-1}}{a} + \frac{(x_1^2 + Dy_1^2)x_{2n-3} + 2Dx_1y_1y_{2n-3}}{a}\right) \\
- & x_{2n-3} \\
= & (x_1^2+1)(x_{2n+1} + x_{2n-1}) - x_{2n-3},
\end{aligned}$$

and

$$\begin{aligned}
& y_{2n+3} \\
= & \frac{2x_1y_1((x_1^2+1)(x_{2n-1} + x_{2n-3}) - x_{2n-5}) + (x_1^2 + Dy_1^2)((x_1^2+1)(y_{2n-1} + y_{2n-5})}{a} \\
= & (x_1^2+1)\left(\frac{2x_1y_1(x_{2n-1} + x_{2n-3}) + (x_1^2 + Dy_1^2)(y_{2n-1} + y_{2n-3})}{a}\right) \\
- & \frac{2x_1y_1x_{2n-5} + (x_1^2 + Dy_1^2)y_{2n-5}}{a} \\
= & (x_1^2+1)\left(\frac{2x_1y_1x_{2n-1} + (x_1^2 + Dy_1^2)y_{2n-1}}{a} + \frac{2x_1y_1x_{2n-3} + (x_1^2 + Dy_1^2)y_{2n-3}}{a}\right) \\
- & y_{2n-3} \\
= & (x_1^2+1)(y_{2n+1} + y_{2n-1}) - y_{2n-3},
\end{aligned}$$

completing the proof.

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On the mean value of the Smarandache LCM function

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Abstract For any positive integer n, the F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$, and let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then $\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s$. The main purpose of this paper is using the elementary methods to study the mean value properties of $\overline{\Omega}(n)SL(n)$, and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, $\overline{\Omega}(n)$ function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of $1,\ 2,\ \cdots,\ k$. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 4$, $SL(9) = 6$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [2] and [3]. For example, Lv Zhongtian [4] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{b_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where b_i $(i = 2, 3, \cdots, k)$ are computable constants.

On the other hand, Chen Jianbin [5] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n \le x} (SL(n) - P(n))^2 = \frac{2}{5} \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

Now we define a new arithmetical function $\overline{\Omega}(n)$ as follows: $\overline{\Omega}(1) = 0$; for $n > 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then $\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s$.

Obviously, for any positive integers m and n, we have $\overline{\Omega}(mn) = \overline{\Omega}(m) + \overline{\Omega}(n)$. That is, $\overline{\Omega}(n)$ is the additive function. The main purpose of this paper is using the elementary methods to study the mean value properties of $\overline{\Omega}(n)SL(n)$, and give a sharper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}\overline{\Omega}(n)SL(n) = \sum_{i=1}^{k}\frac{d_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where d_i $(i = 1,\ 2,\ \cdots,\ k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall use the elementary methods to complete the proof of the theorem.

In fact, for any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime powers, then from [2] we know that

$$SL(n) = \max\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_s^{\alpha_s}\}, \tag{1}$$

and we easily to know that

$$\overline{\Omega}(n) = \alpha_1 p_1 + \alpha_2 p_2 + \cdots + \alpha_s p_s. \tag{2}$$

Now we consider the summation

$$\sum_{n\leq x}\overline{\Omega}(n)SL(n). \tag{3}$$

We separate all integer n in the interval $[1, x]$ into four subsets A, B, C and D as follows:

A: $p \geq \sqrt{n}$ and $n = m \cdot p$;

B: $n^{\frac{1}{3}} < p_1 < p_2 \leq \sqrt{n}$ and $n = m \cdot p_1 \cdot p_2$, where p_i $(i = 1, 2)$ are primes;

C: $n^{\frac{1}{3}} < p \leq \sqrt{n}$ and $n = m \cdot p^2$;

D: otherwise.

It is clear that if $n \in A$, then from (1) we know that $SL(n) = p$, and from (2) we know that $\overline{\Omega}(n) = \overline{\Omega}(m) + p$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [6]) and the Prime Theorem (See Theorem 3.2 of [7]):

$$\pi(x) = \sum_{i=1}^{k}\frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}\sum_{n\in A}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp\le x\\ m<p}}\left(\overline{\Omega}(m)+p\right)p=\sum_{\substack{mp\le x\\ m<p}}p^2+\sum_{\substack{mp\le x\\ m<p}}\left(\overline{\Omega}(m)p\right)\\ &= \sum_{m\le\sqrt{x}}\sum_{m<p\le\frac{x}{m}}p^2+O(x^2)\\ &= \sum_{m\le\sqrt{x}}\left[\pi\left(\frac{x}{m}\right)\frac{x^2}{m^2}-\pi(m)m^2-2\int_m^{\frac{x}{m}}\pi(t)tdt\right]+O(x^2)\\ &= \frac{1}{3}\zeta(3)\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{b_ix^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),\end{aligned}\tag{4}$$

where $\zeta(s)$ is the Riemann zeta-function, and b_i $(i=2,3,\cdots,k)$ are computable constants.

Similarly, if $n\in B$, then we have $SL(n)=p_2$ and $\overline{\Omega}(n)=\overline{\Omega}(m)+p_1+p_2$. So

$$\begin{aligned}\sum_{n\in B}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}\left(\overline{\Omega}(m)+p_1+p_2\right)p_2=\sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}p_2^2+O\left(\sum_{\substack{mp_1p_2\le x\\ m<p_1<p_2}}p_1p_2\right)\\ &= \sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\sum_{p_1<p_2\le\frac{x}{p_1m}}p_2^2+O(x^2)\\ &= \sum_{m\le x^{\frac{1}{3}}}\sum_{m<p_1\le\sqrt{\frac{x}{m}}}\left[\pi\left(\frac{x}{p_1m}\right)\frac{x^2}{p_1^2m^2}-\pi(p_1)p_1^2-2\int_{p_1}^{\frac{x}{p_1m}}\pi(t)tdt\right]\\ &\quad +O(x^2)\\ &= \sum_{i=1}^{k}\frac{c_ix^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),\end{aligned}\tag{5}$$

where c_i $(i=1,2,\cdots,k)$ are computable constants.

Now we estimate the error terms in set C. Using the same method of proving (4), we have $SL(n)=p^2$ and $\overline{\Omega}(n)=\overline{\Omega}(m)+2p$, so

$$\begin{aligned}\sum_{n\in C}\overline{\Omega}(n)SL(n) &= \sum_{\substack{mp^2\le x\\ m<p}}\left(\overline{\Omega}(m)+2p\right)p^2=2\sum_{\substack{mp^2\le x\\ m<p}}p^3+\sum_{\substack{mp^2\le x\\ m<p}}\left(\overline{\Omega}(m)p^2\right)\\ &= 2\sum_{m\le x^{\frac{1}{3}}}\sum_{m<p\le\sqrt{\frac{x}{m}}}p^3+O(x^{\frac{3}{2}})\\ &= O(x^2).\end{aligned}\tag{6}$$

Finally, we estimate the error terms in set D. For any integer $n\in D$, if $SL(n)=p$ then

$p \le \sqrt{n}$; if $SL(n) = p^2$, then $p \le n^{\frac{1}{3}}$; or $SL(n) = p^\alpha, \alpha \ge 3$. So we have

$$\begin{aligned}\sum_{n\in D} \overline{\Omega}(n)SL(n) &\ll \sum_{\substack{mp\le x\\ p\le m}} (\overline{\Omega}(m)+p)p + \sum_{\substack{mp^2\le x\\ p\le m}} (\overline{\Omega}(m)+2p)p \\ &+ \sum_{\substack{mp^\alpha\le x\\ p\le x^{\frac{1}{3}},\alpha\ge 3}} (\overline{\Omega}(m)\alpha p)p^\alpha \ll \frac{x^2}{\ln x}. \end{aligned} \tag{7}$$

Combining (4), (5), (6) and (7) we may immediately obtain the asymptotic formula

$$\begin{aligned}\sum_{n\le x} \overline{\Omega}(n)SL(n) &= \sum_{n\in A} \overline{\Omega}(n)SL(n) + \sum_{n\in B} \overline{\Omega}(n)SL(n) \\ &\quad + \sum_{n\in C} \overline{\Omega}(n)SL(n) + \sum_{n\in D} \overline{\Omega}(n)SL(n) \\ &= \sum_{i=1}^{k} \frac{d_i x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),\end{aligned}$$

where d_i $(i = 1, 2, \cdots, k)$ are computable constants.

This completes the proof of Theorem.

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Four problems related to the Pseudo-Smarandache-Squarefree function

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Abstract For any positive integer n, the Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is, $Zw(n) = \min\{m : \ m \in N, \ n \mid m^n\}$. In reference [2], Felice Russo proposed many problems and conjectures related to the Pseudo-Smarandache-Squarefree function $Zw(n)$. The main purpose of this paper is using the elementary methods to study several problems in [2], and four of them are solved.

Keywords Pseudo-Smarandache-Squarefree function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the famous Pseudo-Smarandache-Squarefree function $Zw(n)$ is defined as the smallest positive integer m such that m^n is divisible by n. That is,

$$Zw(n) = \min\{m : \ m \in N, \ n \mid m^n\}.$$

It is easy to see that if $n > 1$, then

$$Zw(n) = \prod_{p|n} p,$$

where $\prod_{p|n}$ denotes the product over all different prime divisors of n (see reference [1]). From this formula, we can easily get the value of $Zw(n)$. For example, $Zw(1) = 1, \ Zw(2) = 2, \ Zw(3) = 3, \ Zw(4) = 2, \ Zw(5) = 5, \ Zw(6) = 6, \ Zw(7) = 7, \ Zw(8) = 2, \ Zw(9) = 3, \ Zw(10) = 10, \ \cdots$. In fact if n is a square-free number, then $Zw(n) = n$. In reference [2], Felice Russo studied the properties of $Zw(n)$, and proposed the following four problems:

Problem 1. Find all the values of n such that $Zw(n) = Zw(n+1) \cdot Zw(n+2)$.

Problem 2. Solve the equation $Zw(n) \cdot Zw(n+1) = Zw(n+2)$.

Problem 3. Solve the equation $Zw(n) \cdot Zw(n+1) = Zw(n+2) \cdot Zw(n+3)$.

Problem 4. Find all the values of n such that $S(n) = Zw(n)$, where $S(n)$ is the Smarandache function.

The main purpose of this paper is using the elementary methods to study these four problems, and solved them completely. That is, we shall prove the following conclusions:

Theorem 1. The following three equations have no positive integer solution.

$$Zw(n) = Zw(n+1) \cdot Zw(n+2); \tag{1}$$

$$Zw(n) \cdot Zw(n+1) = Zw(n+2); \tag{2}$$

$$Zw(n) \cdot Zw(n+1) = Zw(n+2) \cdot Zw(n+3). \tag{3}$$

Theorem 2. There exist infinite positive integers n such that the equation $S(n) = Zw(n)$, where $S(n)$ is the Smarandache function defined by $S(n) = \min\{k : \ k \in N, \ n \mid k!\}$.

§2. Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove that the equation $Zw(n) = Zw(n+1) \cdot Zw(n+2)$ has no positive integer solution. It is clear that $n = 1$ is not a solution of this equation. In fact if $n = 1$, then

$$1 = Zw(1) \neq 2 \cdot 3 = Zw(2) \cdot Zw(3).$$

If $n > 1$, suppose that the equation (1) has one positive integer solution $n = n_0$, then

$$Zw(n_0) = Zw(n_0+1) \cdot Zw(n_0+2).$$

For any prime divisor p of $n = n_0$, it is clear that $p \mid Zw(n_0)$.

From $Zw(n_0) = Zw(n_0+1) \cdot Zw(n_0+2)$

we deduce that $p \mid Zw(n_0+1) \cdot Zw(n_0+2)$.

That is, $p \mid Zw(n_0+1)$ or $p \mid Zw(n_0+2)$.

(a) If $p \mid Zw(n_0+1)$, then $p \mid (n_0+1)$, combining $p \mid n_0$ and $p \mid (n_0+1)$ we get $p \mid n_0 + 1 - n_0 = 1$, this is a contradiction.

(b) If $p \mid Zw(n_0+2)$, then $p \mid (n_0+2)$, combining $p \mid n_0$ and $p \mid (n_0+2)$ we deduce that $p \mid n_0 + 2 - n_0 = 2$, then we get $n_0 = p = 2$.

From equation (1) we have

$$2 = Zw(2) \neq 3 \cdot 2 = Zw(3) \cdot Zw(4),$$

It is not possible. So the equation (1) has no positive integer solution.

Using the similar method as in proving problem 1, we find that the equation (2) and equation (3) also have no positive integer solution. This completes the proof of Theorem 1.

In order to prove Theorem 2, we need some important properties of the Smarandache function $S(n)$, which we mentioned as the following:

Lemma 1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$S(n) = \max_{1 \leq i \leq k} \{S(p_i^{\alpha_i})\}.$$

Proof. See reference [3].

Lemma 2. If p be a prime, then $S(p^k) \leq kp$; If $k < p$, then $S(p^k) = kp$, where k is any positive integer.

Proof. See reference [4].

Now we use these two simple lemmas to prove our Theorem 2. It's clear that all prime p are the solutions of the equation $S(n) = Zw(n)$. So there are infinite positive integers n satisfying the equation $S(n) = Zw(n)$.

Now we construct infinite composite numbers n satisfying the equation $S(n) = Zw(n)$, let $n = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k^{\alpha_k}$, where p_i are the different primes, and $p_k > \alpha_k = p_1 p_2 \cdot \cdots p_{k-1}$. This time, from the definition of $S(n)$ and $Zw(n)$ we have $S(n) = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k$ and $Zw(n) = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k$. So all composite numbers $n = p_1 \cdot p_2 \cdots p_{k-1} \cdot p_k^{\alpha_k}$ (where p_i are the different primes, and $p_k > \alpha_k = p_1 p_2 \cdot \cdots p_{k-1}$) satisfying the equation $S(n) = Zw(n)$.

Note that k be any positive integer and there are infinite primes, so there are infinite composite numbers n satisfying the equation $S(n) = Zw(n)$.

This complete the proof of Theorem 2.

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On the pseudo Smarandache function

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Abstract For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is, $Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary and analytic methods to study the mean value properties of $\ln Z(n)$, and give an interesting asymptotic formula for it.

Keywords Pseudo Smarandache function, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is,

$$Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are:

$$Z(1) = 1,\ Z(2) = 3,\ Z(3) = 2,\ Z(4) = 7,\ Z(5) = 4,\ Z(6) = 3,\ Z(7) = 6,\ Z(8) = 15,$$
$$Z(9) = 8, Z(10) = 4,\ Z(11) = 10,\ Z(12) = 8,\ Z(13) = 12,\ Z(14) = 7,\ Z(15) = 5,$$
$$Z(16) = 31,\ Z(17) = 16,\ Z(18) = 8,\ Z(19) = 18,\ Z(20) = 15,\ \cdots\cdots .$$

This function was introduced by David Gorski in reference [1], where he studied the elementary properties of $Z(n)$, and obtained a series interesting results. Some of them are as follows:

If $p \geq 2$ be a prime, then $Z(p) = p - 1$;

If $n = 2^k$, then $Z(n) = 2^{k+1} - 1$.

Let p be an odd prime, then $Z(2p) = p$, if $p \equiv 3(\mathrm{mod}4)$; $Z(2p) = p - 1$, if $p \equiv 1(\mathrm{mod}4)$.

For any odd prime p with $p \mid n$ and $n \neq p$, $Z(n) \geq p - 1$.

The other contents related to the pseudo Smarandache function can also be found in references [2], [3] and [4]. In this paper, we consider the mean value properties of $\ln Z(n)$. About this problem, it seems that none had studied it yet, at least we have not seen any related results before. The main purpose of this paper is using the elementary and analytic methods

study this problem, and give an interesting asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} \ln Z(n) = x\ln x + O(x).$$

Whether there exists an asymptotic formula for the mean value

$$\sum_{n\leq x} Z(n) \quad \text{or} \quad \sum_{n\leq x} \frac{1}{Z(n)}$$

are two open problems.

§2. Proof of the theorem

In this section, we shall complete the proof of Theorem. First we need the following simple conclusion:

Lemma. For all real number $x > 1$, we have the asymptotic formula

$$\sum_{p\leq x} \frac{\ln p}{p} = \ln x + O(1),$$

where $\sum_{p\leq x}$ denotes the summation over all prime p with $2 \leq p \leq x$.

Proof. See Theorem 4.10 of reference [5].

Using this Lemma we can prove our Theorem easily. In fact for any positive integer $n > 1$, note that $n \mid \dfrac{2n(2n-1)}{2}$, from the definition of $Z(n)$ we know that $Z(n) \leq 2n-1$. So by the Euler's summation formula we can get

$$\sum_{n\leq x} \ln Z(n) \leq \sum_{n\leq x} \ln(2n-1) \leq x\ln x + O(x). \tag{1}$$

Now let A denotes the set of all square-full numbers n (That is, if $p \mid n$, then $p^2 \mid n$) in the interval $[1,\ x]$. Then we have

$$\sum_{n\leq x} \ln Z(n) = \sum_{\substack{n\leq x\\ n\in A}} \ln Z(n) + \sum_{\substack{n\leq x\\ n\notin A}} \ln Z(n). \tag{2}$$

From reference [6] we know that

$$\sum_{\substack{n\leq x\\ n\in A}} 1 = \frac{\zeta(\frac{3}{2})}{\zeta(3)} x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)} x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}} \exp\left(-C\log^{\frac{3}{5}} x(\log\log x)^{-\frac{1}{5}}\right)\right),$$

where $C > 0$ is a constant. By this estimate and note that $\ln Z(n) \leq \ln(2n)$, we may get

$$\sum_{\substack{n\leq x\\ n\in A}} \ln Z(n) \ll \sqrt{x}\cdot \ln x. \tag{3}$$

If $n \notin A$, then $n = 1$ or there exists at least one prime p with $p \mid n$ and $p^2 \nmid n$. So from Lemma we have

$$\begin{aligned}\sum_{\substack{n\le x\\ n\notin A}} \ln Z(n) &= \sum_{\substack{np\le x\\ (n,\ p)=1}} \ln Z(np) \ge \sum_{p\le x} \sum_{\substack{n\le \frac{x}{p}\\ (n,\ p)=1}} \ln(p-1)\\ &= \sum_{p\le x}\left[\frac{x}{p} - \frac{x}{p^2} + O(1)\right]\cdot \ln(p-1)\\ &= x\cdot\sum_{p\le x}\frac{\ln p}{p} - x\cdot\sum_{p\le x}\frac{\ln p}{p^2} + O(x)\\ &= x\cdot\ln x + O(x). \qquad (4)\end{aligned}$$

Combining (1), (2), (3) and (4) we may immediately get the asymptotic formula

$$\sum_{n\le x}\ln Z(n) = x\ln x + O(x).$$

This completes the proof of Theorem.

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A miscellaneous remark on problems involving Mersenne primes

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23, November 2005

Abstract In this paper, I present some problems involving Mersenne primes for Scientia Magna.

The primes $2^p - 1$ are called Mersenne primes. The set of p is

$$\Gamma = \{\gamma_1 = 2,\ \gamma_2 = 3,\ \gamma_3 = 5,\ 7,\ 13,\ 17,\ 19,\ 31,\ 61,\ 89,\ 107,\ 127, \cdots\}.$$

If $q = 2^p - 1$ is a Mersenne prime, then p is prime (by Cataldi and Fermat) and $n = 2^{p-1}q$ is an even perfect number (Euclides proved that every perfect number is of this type). There is an open problem about Mersenne primes:

An open problem. Are there infinite many Mersenne primes?

The numbers $2^\pi - 1$ (with π prime) are called Mersenne numbers. There are many open problems about Mersenne numbers:

(1) Are there infinitely many composite Mersenne numbers?

(2) Is every Mersenne number square-free?

In this paper, I present the following sequence related with Mersenne primes

$$\omega_1 = 1 + 2,\ \omega_2 = 1 + 2 + 2^2,\ \omega_3 = 1 + 2 + 2^2 + 2^4,\ 1 + 2 + 2^2 + 2^4 + 2^6, \cdots$$

where $\omega_n = \omega_{n-1} + 2^{\gamma_n - 1}$ for all $n > 1$. The prime factorizations of the first terms are: $3,\ 7,\ 23,\ 3 \cdot 29,\ 47 \cdot 89,\ 13 \cdot 31 \cdot 173,\ 3 \cdot 7 \cdot 15803,\ 631 \cdot 1702177, \cdots$.

My problems for Scientia Magna are:

Are there infinitely many primes in the sequence ω_k?

Are there infinitely many composite numbers in the sequence ω_k?

Is every term of the sequence ω_k square-free?

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On the F.Smarandache LCM function and its mean value

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. The main purpose of this paper is using the elementary methods to study the mean value properties of $\ln SL(n)$, and give a sharper asymptotic formula for it.

Keywords F.Smarandache LCM function, mean value, asymptotic formula.

§1. Introduction and Results

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. About the elementary properties of $SL(n)$, some authors had studied it, and obtained some interesting results, see reference [3] and [4]. For example, Murthy [4] showed that if n is a prime, then $SL(n) = S(n)$, where $S(n)$ denotes the Smarandache function, i.e., $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [4] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{1}$$

Le Maohua [5] completely solved this problem, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes, and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Lv Zhongtian [6] obtained the asymptotic formula:

$$\sum_{n \leq x} SL(n) = \frac{\pi^2}{12} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right).$$

In reference [7], Professor Zhang Wenpeng asked us to study the asymptotic properties of $\sum_{n \leq x} \ln SL(n)$. About this problem, it seems that none had studied it, at least we have not

seen related papers before. The main purpose of this paper is using the elementary methods to study this problem, and obtain a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} \ln SL(n) = x\ln x + O(x).$$

Using the same method of proving Theorem 1 we can also give a similar asymptotic formula for the F.Smarandache function $S(n)$. That is, we have the following:

Theorem 2. For any real number $x > 1$, we have

$$\sum_{n\leq x} \ln S(n) = x\ln x + O(x),$$

where $S(n)$ denotes the Smarandache function.

§2. Proof of the theorems

To complete the proof of the theorems, we need the following two important Lemmas.

Lemma 1. For any positive integer $n > 1$, let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ denotes the factorization of n into prime powers, if $\alpha_1 \geq 2, \alpha_2 \geq 2, \cdots \alpha_s \geq 2$, then we call such an integer n as a square-full number. Let $A_2(x)$ denotes the number of all square-full integers not exceeding x, then we have the asymptotic formula

$$A_2(x) = \frac{\zeta(\frac{3}{2})}{\zeta(3)}x^{\frac{1}{2}} + \frac{\zeta(\frac{2}{3})}{\zeta(2)}x^{\frac{1}{3}} + O\left(x^{\frac{1}{6}}\exp\left(-C\log^{\frac{3}{5}}x(\log\log x)^{-\frac{1}{5}}\right)\right), \tag{2}$$

where $C > 0$ is a constant.

Proof. See reference [8].

Lemma 2. Let p be a prime, k be any positive integer. Then for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{\substack{pk\leq x\\(p,\ k)=1}} \ln p = x\ln x + O(x).$$

Proof. From the several different forms of the Prime Theorem (See reference [2], [7] and [9]), we know that

$$\sum_{k\leq x}\frac{\ln p}{p} = \ln x + O(1),$$

$$\sum_{k\leq x}\ln p = x + O\left(\frac{x}{\ln x}\right),$$

and

$$\sum_{k\leq x}\frac{\ln p}{p^2} = D + O\left(\frac{1}{\ln x}\right),$$

where D be an positive constant.

Applying these asymptotic formulas, we have

$$\begin{aligned}\sum_{\substack{pk\le x\\(p,\ k)=1}}\ln p &= \sum_{p\le x}\ln p\sum_{\substack{k\le\frac{x}{p}\\(p,\ k)=1}}1\\ &= \sum_{p\le x}\ln p\left(\frac{x}{p}-\frac{x}{p^2}+O\left(1\right)\right)\\ &= x\sum_{p\le x}\frac{\ln p}{p}-x\sum_{p\le x}\frac{\ln p}{p^2}+O\left(\sum_{p\le x}\ln p\right)\\ &= x\ln x+O\left(x\right).\end{aligned}$$

This proves Lemma 2.

Now, we use these Lemmas to complete the proof of our theorems. Let $U(n)=\sum_{n\le x}\ln SL(n)$. First, we estimate the upper bound of $U(n)$. In fact, from the definition of F.Smarandache LCM function $SL(n)$, we know that for any positive integer n, $SL(n)\le n$ and $\ln SL(n)\le\ln n$, so we have

$$\sum_{n\le x}\ln SL(n)\le\sum_{n\le x}\ln n.$$

Then from the Euler's summation formula (see reference [2]), we may immediately deduce that

$$U(n)\le\sum_{n\le x}\ln n=x\ln x-x+O\left(\ln x\right)=x\ln x+O\left(x\right).\tag{3}$$

Now we estimate the lower bound of $U(n)$. For any positive integer $n>1$, let $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}$ be the factorization of n into prime power, we divide the interval $[1,\ n]$ into two subsets A and B. A denotes the set of all integers in the interval $[1,\ n]$ such that $\alpha_i\ge 2$ $(i=1,\ 2,\ \cdots,\ s)$. That is to say, A denotes the set of all square-full numbers in the interval $[1,\ n]$; B denotes the set of all integers n with $n\in[1,\ n]$ but $n\notin A$. Then we have

$$U(n)=\sum_{\substack{n\le x\\n\in A}}\ln SL(n)+\sum_{\substack{n\le x\\n\in B}}\ln SL(n).$$

From Lemma 1 and the definition of A, we have

$$\sum_{\substack{n\le x\\n\in A}}\ln SL(n)\le\sum_{\substack{n\le x\\n\in A}}\ln n\le\sum_{\substack{n\le x\\n\in A}}\ln x=\ln x\sum_{\substack{n\le x\\n\in A}}1=\ln x\cdot A_2(x)\ll\sqrt{x}\ln x.\tag{4}$$

Now we estimate the summation in set B. Since $SL(n)=\max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_s^{\alpha_s}\}$ (see reference [4]), so for any $n\in B$, there must exist a prime p such that $p|n$ and $p^2\dagger n$. Therefore, from the definition of $SL(n)$ we have $SL(np)\ge p$. Using this estimate we may immediately deduce that

$$\sum_{\substack{n\le x\\n\in B}}\ln SL(n)=\sum_{\substack{np\le x\\(n,\ p)=1}}\ln SL(np)\ge\sum_{\substack{np\le x\\(n,\ p)=1}}\ln p.\tag{5}$$

Then from Lemma 2 and (5), we have

$$\sum_{\substack{n\leq x\\ n\in B}} \ln SL(n) \geq x\ln x + O(x). \tag{6}$$

Combining (3) and (6), we may immediately deduce the asymptotic formula

$$\sum_{n\leq x} \ln SL(n) = x\ln x + O(x).$$

This completes the proof of Theorem 1.

Using the same method of proving Theorem 1, we can also prove Theorem 2.

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A generalized abc-theorem for functions of several variables

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Abstract In this paper we give a generalization of the abc-theorem for functions of several variables.

Keywords *abc*-theorem, Wronskian.

§1. Introduction

Let F be a fixed algebraically closed field of characteristic 0. Let $f(z)$ be a non-constant polynomial with coefficients in F and let $r(f)$ be the number of distinct zeros of f. Then we have the following.

abc-theorem.([3]) Let $a(z),\ b(z),\ c(z)$ be relatively prime polynomials in F and not all constants such that $a+b=c$. Then

$$\max\{\deg(a),\deg(b),\deg(c)\} \leq r(abc)-1.$$

In [9], there is a generalization of the above theorem. The equality $a+b=c$ is replaced by $f_0+\cdots+f_{n+1}=0$ and the functions f_i are polynomials in several variables, relatively prime by pairs. In this paper, we give a similar result for the case of the functions f_i have no common zeros.

Let f is a polynomial in several variables with coefficients in F and f has a factorization

$$f=\prod_{i=1}^{s} p_i^{\alpha_i},$$

where the polynomials p_i are irreducible, distinct, at most one of them is constant, and the $\alpha_i>0$ are integers. Define

$$N_0(f)=\deg\left(\prod_{i=1}^{s} p_i\right).$$

The main theorem is the following:

Theorem 1.1. Let $f_0,\ \cdots,f_{n+1}$ be $n+2$ polynomials in several variables in $F[x_1,\cdots,x_l]$ have no common zero such that $f_0,\cdots,f_n$ are linearly independent. Assume also that

$$f_0+\cdots+f_{n+1}=0. \tag{1}$$

Then

$$\max_{0\le i\le n+1} \deg f_i \le n\left(\sum_{i=0}^{n+1} N_0(f_i) - 1\right).$$

The following corollary is useful for study on the problem of diophantine equations over function fields.

Corollary 1.2. Let $f_0, \cdots, f_{n+1}$ be $n+2$ polynomials in several variables in $F[x_1, \cdots, x_l]$, not all constants and have no common zero such that $f_0^{m_0}, \cdots, f_n^{m_n}$ are linearly independent. Assume also that

$$f_0^{m_0} + \cdots + f_{n+1}^{m_{n+1}} = 0, \tag{2}$$

where $m_0, \cdots, m_{n+1}$ are positive integers. Then

$$\frac{1}{m_0} + \cdots + \frac{1}{m_{n+1}} > \frac{1}{n}.$$

§2. Proof of the main theorem

Let f is a rational function in several variables, we write f in the form:

$$f = \frac{f_1}{f_2},$$

Where $f_1,\ f_2$ are polynomial functions are non-zero and relatively prime on $F[x_1, \cdots, x_l]$. The degree of f, denoted by $\deg f$, is defined to be $\deg f_1 - \deg f_2$.

Let p is a non-constant irreducible polynomial, we write f in the form:

$$f = p^{\alpha}\frac{g_1}{g_2},$$

such that p is not a divisor of $g_1 g_2$, where $g_1,\ g_2$ are polynomials, then α is called the order of f at p and is denoted by μ_f^p. We have the following easily proved properties of μ_f^p.

Lemma 2.1. Let $f,\ g$ be two polynomials and $p \in F[x_1, \cdots, x_l]$ is a non-constant irreducible polynomial, we have

a) $\mu_{f+g}^p \ge \min(\mu_f^p,\ \mu_g^p)$,
b) $\mu_{fg}^p = \mu_f^a + \mu_g^p$,
c) $\mu_{\frac{f}{g}}^p = \mu_f^p - \mu_g^p$.

For Δ is a differential operator of the form

$$\Delta = (\mu_1 \cdots \mu_m)^{-1}\frac{\partial^{\mu_1}}{\partial x_1^{\mu_1}} \cdots \frac{\partial^{\mu_m}}{\partial x_m^{\mu_m}},$$

where $\mu_i \ge 0$ are integers, we denote the rank of Δ by

$$\rho(\Delta) = \sum_{i=1}^{m} \mu_i.$$

Note that in the above definition, whenever μ_i is zero, we admit the following convention: we eliminate μ_i and consider the operator $\dfrac{\partial^{\mu_i}}{\partial x_1^{\mu_i}}$ is an identity operator.

Lemma 2.2. Let φ be a polynomial in several variables satisfy the following $\Delta\varphi \not\equiv 0$, p be a non-constant irreducible polynomial. Then

$$\mu^p_{\Delta\varphi} \geq -\rho(\Delta) + \mu^p_{\varphi}.$$

Proof. Let $\mu^p_{\varphi} = m$, then there exists a polynomial f such that $\varphi = p^m f$. We have

$$\frac{\partial\varphi}{\partial x_i} = p^{m-1}(p\frac{\partial f}{\partial x_i} + mf\frac{\partial p}{\partial x_i}).$$

From this, we have

$$\mu^p_{\frac{\partial\varphi}{\partial x_i}} \geq m - 1.$$

Therefore,

$$\mu^p_{\frac{\partial\varphi}{\partial x_i}} \geq -1 + \mu^p_{\varphi}.$$

From this, we obtain

$$\mu^p_{\Delta\varphi} \geq -\rho(\Delta) + \mu^p_{\varphi}.$$

Given $\Delta_0, \cdots, \Delta_s$, such that $\rho(\Delta_i) \leq i$ and polynomials $h_0, \cdots, h_s$ in $F[x_1, \cdots, x_l]$, a generalized Wronskian has the form

$$W[h_0, \cdots, h_s] = \det|\Delta_i h_j|_{0\leq i,j\leq s}. \tag{3}$$

A well-known result (see [7], [8]) asserts that, if the functions h_i are linearly independent over F, then there exists a generalized Wronskian of the (3), which does not vanish.

Proof of Theorem 1.1. By the hypothesis, $f_0, \cdots, f_n$ are linearly independent, we have there exists a generalized Wronskian W of $f_0, \cdots, f_n$ does not vanish. We set

$$P = \frac{W(f_0, \cdots, f_n)}{f_0 \cdots f_n},$$

$$Q = \frac{f_0 \cdots f_{n+1}}{W(f_0, \cdots, f_n)}.$$

Hence, we have

$$f_{n+1} = PQ. \tag{4}$$

We first prove that

$$\deg Q \leq n\sum_{i=0}^{n+1} N_0(f_i).$$

Suppose that p is a divisor of $f_0 f_1 \cdots f_{n+1}$ and p is a non-constant irreducible polynomial. By the hypothesis, we have there exists ν, $0 \leq \nu \leq n+1$, such that p is not a divisor of f_ν. By the hypothesis, $f_0 + \cdots + f_n + f_{n+1} = 0$, we have

$$\begin{aligned}\mu^p_{\frac{f_0\cdots f_{n+1}}{W(f_0,\cdots,f_n)}} &= \mu^p_{\frac{f_0\cdots f_{\nu-1}f_{\nu+1}\cdots f_{n+1}}{W(f_0,\cdots,f_{\nu-1},f_{\nu+1},\cdots,f_{n+1})}} \\ &= \sum_{j=0}^{n+1}\mu^p_{f_j} - \mu^p_{W(f_0,\cdots,f_{\nu-1},f_{\nu+1},\cdots,f_{n+1})}.\end{aligned}$$

$W(f_0, \cdots, f_{\nu-1}, f_{\nu+1}, \cdots f_{n+1})$ is the sum of following terms

$$\delta\Delta_0 f_{\alpha_0}\Delta_1 f_{\alpha_1}\cdots\Delta_n f_{\alpha_n},$$

where $\alpha_i \in \{0,\cdots,n+1\}\backslash\{\nu\}$, $\delta = \pm 1$. By Lemma 2.1, 2.2 and note that $\rho(\Delta_i) \le i \le n$, we have

$$\begin{aligned}
& \mu^p_{\Delta_0 f_{\alpha_0}\Delta_1 f_{\alpha_1}\cdots\Delta_n f_{\alpha_n}} \\
\ge & \sum_{0\le j\le n+1,\ p/f_j} \mu^p_{f_{\alpha_j}} - n \sum_{0\le j\le n+1,\ p/f_j} 1 \\
= & \sum_{j=0}^{n} \mu^p_{f_{\alpha_j}} - n \sum_{0\le j\le n+1,\ p/f_j} 1 \\
= & \mu^p_{\prod_{j=0}^n f_{\alpha_j}} - n \sum_{0\le j\le n+1,\ p/f_j} 1.
\end{aligned}$$

By Lemma 2.1, we have

$$\mu^p_{W(f_0,\cdots,f_{\nu-1},f_{\nu+1},\cdots,f_{n+1})} \ge \mu^p_{\prod_{j=0}^n f_{\alpha_j}} - n \sum_{0\le j\le n+1,\ p/f_j} 1.$$

Hence

$$\mu^p_{\frac{f_0\cdots f_{n+1}}{W(f_0,\cdots,f_n)}} \le n \sum_{0\le j\le n+1,\ p/f_j} 1.$$

By the definition of degree of a rational function, we have

$$\deg Q \le n\sum_{i=0}^{n+1} N_0(f_i). \tag{5}$$

Next, we will prove that

$$\deg P \le -n.$$

We have the determinant P is a summa of following terms

$$\delta\frac{\Delta_0 f_{\beta_0}\Delta_1 f_{\beta_1}\cdots\Delta_n f_{\beta_n}}{f_{\beta_0}f_{\beta_1}\cdots f_{\beta_n}}.$$

For every term, we have

$$\begin{aligned}
\deg\left(\frac{\Delta_0 f_{\beta_0}\Delta_1 f_{\beta_1}\cdots\Delta_n f_{\beta_n}}{f_{\beta_0}f_{\beta_1}\cdots f_{\beta_n}}\right) &= \deg\left(\frac{\Delta_0 f_{\beta_0}}{f_{\beta_0}}\right) + \deg\left(\frac{\Delta_1 f_{\beta_1}}{f_{\beta_1}}\right) + \cdots + \deg\left(\frac{\Delta_n f_{\beta_n}}{f_{\beta_n}}\right) \\
&\le -\rho(\Delta_0) - \rho(\Delta_1) - \cdots - \rho(\Delta_n).
\end{aligned}$$

Note that, $n \le \rho(\Delta_0) + \cdots + \rho(\Delta_n) \le \dfrac{n(n+1)}{2}$, from this, we have

$$\deg P \le -n. \tag{6}$$

From (4), (5), (6), we have

$$\begin{aligned}
\deg f_{n+1} &= \deg P + \deg Q \\
&\le n\left(\sum_{i=0}^{n+1} N_0(f_i) - 1\right).
\end{aligned}$$

Similar arguments apply to the polynomials $f_0,\ f_1,\ \cdots,\ f_n$, we have

$$\max_{0\le i\le n+1}\deg f_i \le n\left(\sum_{i=0}^{n+1}N_0(f_i)-1\right).$$

This completes the proof.

Proof of Corollary 1.2. By Theorem 1.1, we have

$$\begin{aligned} m_j\deg f_j=\deg f_j^{m_j} &\le n(\sum_{j=0}^{n+1}N_0(f_j^{m_j})-1)\\ &= n(\sum_{j=0}^{n+1}N_0(f_j)-1)\\ &< n\sum_{j=0}^{n+1}\deg f_j. \end{aligned}$$

Then, we have

$$\sum_{j=0}^{n+1}\deg f_j < n\sum_{j=0}^{n+1}\frac{1}{m_j}\sum_{j=0}^{n+1}\deg f_j.$$

From this, we obtain (2). This completes the proof.

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An assessment method for weight of experts at interval judgment[1]

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Abstract This paper proposes an assessment method for weight of experts at interval reciprocal judgment based on support vector domain description (SVDD). In this method, firstly, we give the point-vector decomposition to all the columns of every interval judgment matrix. Secondly, the group information is found using the support vector domain description (SVDD), and several concepts are introduced such as group compatibility, group core vectors and group information contribution ratio. Thirdly, we present a computational method for the contribution ratio of every expert so as to determine the assessment weight of every expert. Finally, the feasibility and validity of the method are shown with one example. Using the method can decrease unilateral influence, extract key information, and evaluate weight of each expert more objectively.

Keywords Group decision making, interval reciprocal judgment, compatibility, support vector domain description, weight.

§1. Introduction

The key of ensuring the quality of group decision making is the determination of the weights of experts in the aggregation process. Therefore, the study on the weights of experts is a research hot point in group decision making [7]-[13].

Currently, the principal problem about the evaluation to weight of experts is without eliminating the disturbance information provided by experts in which the given weight of experts is doubting. We think that it is more scientific for the weight of experts that should beforehand remove the unilateral disturbance information, and then appraise the weight according to the magnitude of contribution to group opinion.

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In addition, though there exist many evaluation methods for the weight of experts, we do not have yet find the evaluation method for the interval judgment. Thus, this paper will propose a more objective method for the weight problem of experts at the form of interval reciprocal judgment.

The approach gives firstly a point-vector decomposition to each judgment matrix using the method in preference [4], and then extract group information based on the technique for SVDD [5]. Finally, determine the contribution ratio of group opinion of each expert.

§2. Some concepts

In this section, we briefly describe two basic definition related interval judgment.

Definition 1.[3][4] Matrix $A = (a_{ij})_{n\times n}$ is called a point judgment matrix, for all a_{ij} $(i,\ j = 1, \cdots, n)$, if the following conditions are satisfied:

(1) $1/9 \le a_{ij} \le 9$;

(2) $a_{ij} = 1/a_{ji}$;

where the each column vector is called a point vector. We Simply call the above judgment way point judgment.

Definition 2.[3][4] Matrix $A = (a_{ij})_{n\times n}$ is called a interval reciprocal judgment matrix, for all a_{ij} $(i, j = 1, \cdots, n)$, if the following conditions are satisfied:

(1) $a_{ij} = [l_{ij}, u_{ij}],\ i \ne j$;

(2) $a_{ij} = [1, 1]$;

(3) $l_{ij} = 1/u_{ji}$;

where $[l_{ij},\ u_{ij}]$ is an interval number, $l_{ij},\ u_{ij}$ represents the lower limit, upper limit of the significance ratio of element i to element j, respectively. Here we take also the scale values 1-9.

In following we simply call interval reciprocal judgment matrix interval judgment matrix.

Suppose there exist n decision objects, and m experts: $E^{(1)},\ E^{(2)},\ \cdots,\ E^{(m)}$. Let the interval judgment matrix provided by $E^{(k)}$ $(k = 1, \cdots, m)$ be

$$A^{(k)} = \begin{pmatrix} [1,1] & [l_{12}^{(k)}, u_{12}^{(k)}] & \cdots & [l_{1n}^{(k)}, u_{1n}^{(k)}] \\ [l_{21}^{(k)}, u_{21}^{(k)}] & [1,1] & \cdots & [l_{2n}^{(k)}, u_{2n}^{(k)}] \\ \cdots & \cdots & \cdots & \cdots \\ [l_{n1}^{(k)}, u_{n1}^{(k)}] & [l_{n2}^{(k)}, u_{n2}^{(k)}] & \cdots & [1,1] \end{pmatrix}. \tag{1}$$

§3. Point-judgment decomposition

In this section,we shall give a point-vector decomposition to interval reciprocal judgment matrix (1). All the elements of each column of interval judgment matrix (1) are interval numbers in addition to those elements that lie on main diagonal.

If choosing randomly one of numbers among the lower limit and the upper limit of each interval number, which can form some $n-$ dimensional point vectors together with the element 1 at the diagonal.

Theorem 1. Interval judgment matrix (1) can generate $2^{n-1} \cdot n$ point vectors by decomposition.

Proof. Since any column elements in interval judgment matrix (1) comprise $n-1$ interval numbers. When taking two limit values at each interval number, we can obtain 2^{n-1} kinds of ways taking numbers. Integrating the element 1 lie on diagonal, then each column can construct 2^{n-1} point vectors.

We note that interval judgment matrix (1) comprises n columns, therefore, interval judgment matrix (1) can produce $2^{n-1} \cdot n$ point vectors(n dimension). For example, if $n = 3$, let

$$A = \begin{pmatrix} [1,1] & [l_{12}, u_{12}] & [l_{13}, u_{13}] \\ [l_{21}, u_{21}] & [1,1] & [l_{23}, u_{23}] \\ [l_{31}, u_{31}] & [l_{32}, u_{32}] & [1,1] \end{pmatrix}, \tag{2}$$

then the first column of matrix A can be decomposed into 4 point vectors:

$$\alpha_1 = [1, l_{21}, l_{31}]^T,\ \alpha_2 = [1, l_{21}, u_{31}]^T,\ \alpha_3 = [1, u_{21}, l_{31}]^T,\ \alpha_4 = [1, u_{21}, u_{31}]^T.$$

So, matrix A can generate $2^{3-1} \cdot 3 = 12$ point vectors altogether.

For an $n \times n$ judgment matrix, if the $2^{n-1} \cdot n$ point vectors are normalized, they will lie on the same cone underside [4]. In fact, the convex combination of these point vectors can form a feasible domain of weight vectors of the decision objects [4]. For a group decision making, each expert can all generate such a feasible domain.

§4. Support vector domain description (SVDD) [6]

In this section, we shall introduce briefly SVDD algorithm. Let a set of data be x_i, $x_i \in R^d (i = 1, \cdots, N)$. The algorithm for SVDD is just the description of data by finding a hyper sphere with minimum radius containing nearly this data.

In order to describe better the data, usually a nonlinear mapping ϕ will be mapped onto a high-dimensional feature space, and a kernel function $k(x, y) = \phi(x) \cdot \phi(y)$ will be introduced to instead of inner product.

In order to eliminate the influence of outliers on the description, the slack variables ξ_i and penalty factor C are introduced. Minimization of radius of the hyper sphere is just finding the solution of the following convex quadratic programming problem:

$$\min R^2 + C \sum_{i=1}^{N} \xi_i, \tag{3}$$

$$s.t.\ (\phi(x_i) - a)^T (\phi(x_i) - a) \le R^2 + \xi_i, \tag{4}$$

where $\xi_i \ge 0 (i = 1, \cdots, N)$, and C is a constant that punishes the samples separated by mistake, R is the radius of the sphere, a (vector) is the center of the hyper sphere. To solve the optimization problem, we set lagrange function as follows:

$$L(R, a, \alpha_i, \gamma_i, \xi_i) = \sum_{i=1}^{N} \alpha_i \left[||\phi(x_i)||^2 - 2a \cdot \phi(x_i) + ||a||^2 - R^2 - \xi_i \right] + R^2 + C \sum_{i=1}^{N} \xi_i - \sum_{i=1}^{N} \gamma_i \xi_i, \tag{5}$$

where $\alpha_i \geq 0, \gamma_i \geq 0$.

By **KKT** condition, the dual programming of above convex quadratic programming [6] is

$$\max \sum_{i=1}^{N} \alpha_i k(x_i, x_i) - \sum_{i=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_j k(x_i, x_j), \tag{6}$$

$$s.t. \ \sum_{i=1}^{N} \alpha_i = 1, \ 0 \leq \alpha_i \leq C. \tag{7}$$

We will often have that $\alpha_i = 0$ for most x_i, all the x_i for which $\alpha_i \neq 0$ are called the support vectors. And the center of the sphere can be expressed as

$$a = \sum_{i=1}^{N} \alpha_i \phi(x_i), \ 0 < \alpha_i < C. \tag{8}$$

For some x_j for which $\alpha_j \neq 0$, sphere radius is

$$R^2 = ||\phi(x_j) - a||^2 = k(x_j, x_j) - 2 \sum_{i=1}^{N} \alpha_i k(x_i, x_j) + \sum_{i=1}^{N} \sum_{t=1}^{N} \alpha_i \alpha_t k(x_i, x_t), \tag{9}$$

$$||z - a||^2 = (z \cdot z) - 2 \sum_{i=1}^{N} \alpha_i k(x_i, z) + \sum_{i=1}^{N} \sum_{i=1}^{N} \alpha_i \alpha_j k(x_i, x_j) \leq R^2. \tag{10}$$

Note. All the test vectors will satisfy equation (10) if $C \geq 1$.

§5. Determination method for contribution ratio of group information

From the discussion in section 2 we know that if judgment matrices are given at the form (1), the point-vector decomposition can all be performed.

By Theorem 1, we can construct $2^{n-1} \cdot n$ point vectors from each judgment matrix by the decomposition. Each point vector represents judgment information corresponding expert.

We hope that the weight of each expert can be evaluated in terms of the magnitude of the his (or her) contribution ratio to group opinion. Generally, all the point vectors of each expert comprise either personality or group property.

Our concern is how to extract group information. We think that group opinion is just the nearer opinion among all the experts, that is the opinion indicated by the point vectors with smaller distance among all the point vectors of all the experts.

By reference [4], the convex domain formed by the convex combination of all the point vectors of each expert can be seen as the feasible domain of the weight vector of order alternatives. It is more reasonable that the overlapping domain of feasible domains of all the experts is viewed as the representative domain of group information. However, to find precisely the overlapping domain is rather difficult. Therefore, it is needful to find an approximate method for convenience in computation.

This paper attempts to train a minimum enclosing sphere respectively by the $2^{n-1} \cdot n$ point vectors generated by each expert using the method for SVDD. And then we shall determine the evaluation weight of each expert based on the information provided by the point vectors in public overlapping domain of all the enclosing sphere.

5.1. Determination of penalty parameter [5]

The key in using SVDD to train sample is the choice of parameter. For each judgment matrix, $2^{n-1} \cdot n$ point vectors contain likely outliers (or noises). However, these noises are likely group information with significant value for whole group. Therefore, we shall take parameter $C = 1$ when using SVDD to describe the point vectors of each expert so as to all the point vectors can be contained in the trained sphere.

5.2. Choice of kernel function [6]

Choice of kernel function must be done in using SVDD. We only consider Gauss kernel function as follows

$$k(x, y) = exp[-||x - y||^2/s^2].$$

It is well known that the enclosing sphere generated by Gauss kernel function is tighter if width parameter s is set a smaller value, and the domain enclosed by the sphere is smaller than the feasible domain formed by the convex combination of point vectors.

By reference [5], we know the enclosing sphere generated can generate a sphere surface as the distribution of the point vectors when $s = 1$. Such an enclosing sphere is too tight though it can reduce redundant space in the sphere.However, when $s = 5$, the enclosing sphere generated is a ellipsoid, which is larger than the domain formed by the convex combination of point vectors.

Therefore, in order to more approximate to the ideal feasible domain, we let $s = 2$ in this paper.

5.3. Concepts and computation method

In this section, we shall propose some concepts related to training spheres and the computation method for group information contribution ratio. Let the least enclosing sphere be S_k generated by $2^{n-1} \cdot n$ point vectors from the judgment matrix $A^{(k)}$ provided by expert $E^{(k)}$, and the sphere center be a_k, radius be $R_k (k = 1, \cdots, m)$. Suppose $S_i \bigcap S_j$ represents the common overlapping domain of $S_i$and S_j.

Definition 3. If there exist at least a point vector $V_i \in S_i$, and a point vector $V_j \in S_j$, s.t. $V_i \in S_i \bigcap S_j$ and $V_j \in S_i \bigcap S_j$, then call $S_i \bigcap S_j \neq \phi (i, j = 1, \cdots, m,\ i \neq j)$.

Definition 4. If the $2^{n-1} \cdot n$ point vectors generated by each judgment matrix are completely same, then call the group completely compatible, or we think the group is of complete compatibility.

Note. The case of complete compatibility is impossible to appear in general, usually, part compatibility exists.

Definition 5. If $S_i \bigcap S_j \neq \phi (i, j = 1, \cdots, m, i \neq j)$,then we think that experts $E^{(i)}$ and $E^{(j)}$ exist compatibility.

Definition 6. If $\bar{S} = S_1 \bigcap S_2 \bigcap \cdots \bigcap S_m \neq \phi$, then we say that the expert group exists group compatibility (m is the number of experts).

Definition 7. If $\bar{S} = S_1 \bigcap S_2 \bigcap \cdots \bigcap S_m \neq \phi$, then call point vectors in $\bar{S} = S_1 \bigcap S_2 \bigcap \cdots \bigcap S_m$ group core vectors.

Definition 8. If $\bar{S} = S_1 \bigcap S_2 \bigcap \cdots \bigcap S_m \neq \phi$, suppose $\bar{S}$ contains M point vectors, where r_k point vectors belong to $S_k (k = 1, \cdots, m)$, then call $\omega_k = r_k/M$ group information contribution ratio of $E^{(k)}$, or weight. The obtained vector $w = (\omega_1, \omega_2, \cdots, \omega_m)^T$ is called weight vector of the group experts.

Note. Group opinion is determined completely by the core vectors, and those points outside the minimum sphere are viewed as noise points for the group, therefore, the method has the efficacy de-noising.

§6. Algorithm

The steps of evaluation method are follows:

(1) Perform a point-vector decomposition to each judgment matrix, find all the point vectors of each judgment matrix;

(2) Carry out a training using SVDD, find the least enclosing sphere of each expert;

(3) Judge whether the group is of compatibility;

(4) If the group compatibility exists, then calculate the group information contribution ratio of each expert, i.e. weight.

§7. Example

Suppose interval reciprocal judgment matrices provided by 3 experts for 4 evaluation alternatives are as follows:

$$A^{(1)} = \begin{pmatrix} [1,1] & [2,3] & [4,6] & [7,8] \\ [\frac{1}{3},\frac{1}{2}] & [1,1] & [2,3] & [3,4] \\ [\frac{1}{6},\frac{1}{4}] & [\frac{1}{3},\frac{1}{2}] & [1,1] & [2,3] \\ [\frac{1}{8},\frac{1}{7}] & [\frac{1}{4},\frac{1}{3}] & [\frac{1}{3},\frac{1}{2}] & [1,1] \end{pmatrix},$$

$$A^{(2)} = \begin{pmatrix} [1,1] & [2,4] & [4,5] & [6,7] \\ [\frac{1}{4},\frac{1}{2}] & [1,1] & [2,3] & [2,3] \\ [\frac{1}{5},\frac{1}{4}] & [\frac{1}{3},\frac{1}{2}] & [1,1] & [2,3] \\ [\frac{1}{7},\frac{1}{6}] & [\frac{1}{3},\frac{1}{2}] & [\frac{1}{3},\frac{1}{2}] & [1,1] \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} [1,1] & [3,4] & [3,5] & [6,8] \\ [\frac{1}{4},\frac{1}{3}] & [1,1] & [2,3] & [2,3] \\ [\frac{1}{5},\frac{1}{3}] & [\frac{1}{3},\frac{1}{2}] & [1,1] & [3,4] \\ [\frac{1}{8},\frac{1}{6}] & [\frac{1}{3},\frac{1}{2}] & [\frac{1}{4},\frac{1}{3}] & [1,1] \end{pmatrix}.$$

By the steps in section 5, we can get 32 point vectors from each column of each judgment matrix by the decomposition, and 96 point vectors can generated from 3 judgment matrix

altogether. Here Gauss kernel ($s = 2$) is chosen. After the training of SVDD, we find $\bar{S} \neq \phi$, which means that the group exists compatibility. The training result tells us that $\bar{S}$ contains 64 group core vectors, i.e. $M = 64$, and $r_1 = 30,\ r_2 = 18,\ r_3 = 16$. Then $\omega_1 = 30/64,\ \omega_2 = 18/64,\ \omega_3 = 16/64$.

Therefore, the obtained weight vector is

$$w = [0.469, 0.281, 0.250]^T.$$

Remark.

(1) The core vectors in $\bar{S}$ can gradually decrease as the increase of the number of experts, and non compatibility can likely occurs, which accords with fact. Because the more expert, the more difficult keeping the compatibility. In this case, we can set 0 to the weight value of the experts with weaker compatibility.

(2) When the number of judgment object is larger, the number of point vectors is more, In this case, we can draw randomly a sample in each point vector set, and then use the method again.

§8. Conclusion

The paper is mainly applying SVDD to adverse judgment problem in group decision making, and gave a more objective evaluation method for the evaluation quality of expert at interval reciprocal judgment. The contribution in this paper is mainly two aspects as follows:

(1) Gave an evaluation method for expert weight aiming at interval reciprocal judgment by decomposing the interval judgment into point judgment.

(2) The technology for SVDD was successfully applied to the group decision making at interval reciprocal judgment, which can eliminate disturbance information and find group information, then the evaluation quality of each expert is more fairly judged.

Farther work should consider the application of SVDD to the other group decision making.

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On the F.Smarandache LCM function $SL(n)$

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Abstract For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ is defined as the smallest positive integer k such that $n \mid [1, 2, \cdots, k]$, where $[1, 2, \cdots, k]$ denotes the least common multiple of $1, 2, \cdots, k$. The main purpose of this paper is using the elementary methods to study the mean value distribution property of $(P(n) - p(n))SL(n)$, and give an interesting asymptotic formula for it.

Keywords $SL(n)$ function, mean value distribution, asymptotic formula.

§1. Introduction and Result

For any positive integer n, the famous F.Smarandache LCM function $SL(n)$ defined as the smallest positive integer k such that $n \mid [1,\ 2,\ \cdots,\ k]$, where $[1,\ 2,\ \cdots,\ k]$ denotes the least common multiple of 1, 2, $\cdots$, k. For example, the first few values of $SL(n)$ are $SL(1) = 1$, $SL(2) = 2$, $SL(3) = 3$, $SL(4) = 4$, $SL(5) = 5$, $SL(6) = 3$, $SL(7) = 7$, $SL(8) = 8$, $SL(9) = 9$, $SL(10) = 5$, $SL(11) = 11$, $SL(12) = 4$, $SL(13) = 13$, $SL(14) = 7$, $SL(15) = 5$, $\cdots$. From the definition of $SL(n)$ we can easily deduce that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL(n) = \max\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}. \tag{1}$$

About the elementary properties of $SL(n)$, many people had studied it, and obtained some interesting results, see references [1], [2] and [3]. For example, Murthy [1] porved that if n be a prime, then $SL(n) = S(n)$, where $S(n)$ be the F.Smarandache function. That is, $S(n) = \min\{m :\ n|m!,\ m \in N\}$. Simultaneously, Murthy [1] also proposed the following problem:

$$SL(n) = S(n), \quad S(n) \neq n\ ? \tag{2}$$

Le Maohua [2] solved this problem completely, and proved the following conclusion:

Every positive integer n satisfying (1) can be expressed as

$$n = 12 \quad \texttt{or} \quad n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} p,$$

where p_1, p_2, $\cdots$, p_r, p are distinct primes and α_1, α_2, $\cdots$, α_r are positive integers satisfying $p > p_i^{\alpha_i}$, $i = 1, 2, \cdots, r$.

Zhongtian Lv [3] studied the mean value properties of $SL(n)$, and proved that for any fixed positive integer k and any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x} SL(n) = \frac{\pi^2}{12}\cdot\frac{x^2}{\ln x} + \sum_{i=2}^{k}\frac{c_i\cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where c_i $(i = 2, 3, \cdots, k)$ are computable constants.

Jianbin Chen [4] studied the value distribution properties of $SL(n)$, and proved that for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(SL(n) - P(n))^2 = \frac{2}{5}\cdot\zeta\left(\frac{5}{2}\right)\cdot\frac{x^{\frac{5}{2}}}{\ln x} + O\left(\frac{x^{\frac{5}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and $P(n)$ denotes the largest prime divisor of n.

Xiaoyan Li [5] studied the mean value properties of $P(n)SL(n)$ and $p(n)SL(n)$, and give two sharper asymptotic formulas for them, where $p(n)$ denotes the smallest prime divisor of n.

Yanrong Xue [6] defined another new function $SL^*(n)$ as follows: $SL^*(1) = 1$, and if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$ be the factorization of n into primes powers, then

$$SL^*(n) = \min\{p_1^{\alpha_1},\ p_2^{\alpha_2},\ \cdots,\ p_r^{\alpha_r}\}, \tag{3}$$

where $p_1 < p_2 < \cdots < p_r$ are primes.

It is clear that function $SL^*(n)$ is the dual function of $SL(n)$. So it has close relationship with $SL(n)$. About its elementary property of the function $SL^*(n)$, Yanrong Xue [6] proved the following conclusion:

For any positive integer n, there is no any positive integer $n > 1$ such that

$$\sum_{d|n}\frac{1}{SL^*(d)}$$

is an positive integer, where $\sum_{d|n}$ denotes the summation over all positive divisors of n.

In this paper, we shall study the value distribution properties of $(P(n) - p(n))SL(n)$, and give a sharper asymptotic formula for it. That is, we shall prove the following:

Theorem. For any real number $x > 1$ and any positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(P(n) - p(n))SL(n) = \zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

§2. Proof of the theorem

In this section, we shall complete the proof of the theorem directly. For any positive integer $n > 1$, we consider the following cases:

A: $n = n_1\cdot p$, $n_1 \leq p$, and $SL(n) = p$;

B: $n = n_2 \cdot p$, $n_2 > p$, and $SL(n) = p$;

C: $n = m \cdot p^{\alpha}$, $\alpha \geq 2$, and $SL(n) = p^{\alpha}$;

Now, for any positive integer $n > 1$, we consider the summation:

$$\sum_{n \leq x} (P(n) - p(n)) SL(n).$$

It is clear that if $n \in A$, then from (1) we know that $SL(n) = p$. Therefore, by the Abel's summation formula (See Theorem 4.2 of [7]) and the Prime Theorem (See Theorem 3.2 of [8]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1, 2, \ldots, k)$ are computable constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \in A}} (P(n) - p(n)) SL(n) &= \sum_{\substack{n \leq x \\ n = n_1 \cdot p, n_1 \leq p \\ SL(n) = p}} (P(n) - p(n)) SL(n) \\
&= \sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} (P(n_1 \cdot p) - p(n_1 \cdot p)) p \\
&= \sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} (p - p(n_1)) p \\
&= \sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} p^2 - \sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} p(n_1) p, \qquad (4)
\end{aligned}$$

while

$$\begin{aligned}
\sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} p^2 &= \sum_{n_1 \leq \sqrt{x}} \left[\frac{x^2}{n_1^2} \pi\left(\frac{x}{n_1}\right) - \int_{n_1}^{\frac{x}{n_1}} 2y\pi(y) dy + O\left(n_1^3\right) \right] \\
&= \sum_{n_1 \leq \sqrt{x}} \left[\frac{x^3}{n_1^3} \sum_{i=1}^{k} \frac{b_i}{\ln^i \frac{x}{n_1}} + O\left(\frac{x^3}{n_1^3 \cdot \ln^{k+1} \frac{x}{n_1}}\right) \right] \\
&= \zeta(3) \cdot x^3 \cdot \sum_{i=1}^{k} \frac{b_i}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (5)
\end{aligned}$$

where $\zeta(s)$ is the Riemann zeta-function, $b_1 = \dfrac{1}{3}$, b_i $(i = 2, 3, \cdots, k)$ are computable constants.

Note that $p(n_1) \leq n_1$, we have

$$\begin{aligned}
\sum_{n_1 \leq \sqrt{x}} \sum_{n_1 \leq p \leq \frac{x}{n_1}} p(n_1) p &= \sum_{n_1 \leq \sqrt{x}} p(n_1) \sum_{n_1 \leq p \leq \frac{x}{n_1}} p \\
&= \sum_{n_1 \leq \sqrt{x}} p(n_1) \left[\frac{x}{n_1} \pi\left(\frac{x}{n_1}\right) - \int_{n_1}^{\frac{x}{n_1}} \pi(y) dy + O\left(n_1^2\right) \right] \\
&\ll \sum_{n_1 \leq \sqrt{x}} p(n_1) \cdot \frac{x^2}{n_1^2 \ln x} \ll \sum_{n_1 \leq \sqrt{x}} \frac{x^2}{n_1 \ln x} = O(x^2). \qquad (6)
\end{aligned}$$

From (4), (5)and (6) we have

$$\sum_{\substack{n\le x\\ n\in A}}(P(n)-p(n))SL(n)=\zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right), \tag{7}$$

where $b_1=\dfrac{1}{3}$, b_i $(i=2,3,\cdots,k)$ are computable constants.

If $n\in B$, $SL(n)=p$, then by the Abel's summation formula and the Prime Theorem, we can deduce the following:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in B}}(P(n)-p(n))SL(n) &= \sum_{\substack{n\le x\\ n=n_2\cdot p,n_2>p\\ SL(n)=p}}(P(n)-p(n))SL(n)\\ &= \sum_{\substack{n_2\cdot p\le x\\ n_2>p}}(p-p(n_2))p\\ &\ll \sum_{\substack{n_2\cdot p\le x\\ n_2>p}}p^2=\sum_{p<\sqrt{x}}\sum_{p<n_2\le\frac{x}{p}}p^2\\ &< \sum_{p<\sqrt{x}}\frac{x}{p}\cdot p^2=\sum_{p<\sqrt{x}}x\cdot p\\ &= x\sum_{p<\sqrt{x}}p\ll x^2.\end{aligned} \tag{8}$$

If $n\in C$, then $SL(n)=p^\alpha$, $\alpha\ge 2$. Therefore, using the Abel's summation formula and the Prime Theorem, we can obtain:

$$\begin{aligned}\sum_{\substack{n\le x\\ n\in C}}(P(n)-p(n))SL(n) &= \sum_{\substack{n\le x\\ n=m\cdot p^\alpha,\alpha\ge 2\\ SL(n)=p^\alpha}}(P(n)-p(n))SL(n)\\ &= \sum_{\substack{m\cdot p^\alpha\le x\\ \alpha\ge 2}}(P(m\cdot p^\alpha)-p(m\cdot p^\alpha))p^\alpha\\ &\ll \sum_{\substack{m\cdot p^\alpha\le x\\ \alpha\ge 2}}p^{2\alpha}\ll\sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}\sum_{m\le\frac{x}{p^\alpha}}p^{2\alpha}\\ &\ll \sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}\frac{x}{p^\alpha}\cdot p^{2\alpha}=\sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}x\cdot p^\alpha\\ &= x\sum_{\substack{p^\alpha\le x\\ \alpha\ge 2}}p^\alpha=x\sum_{\substack{p\le x^{\frac{1}{\alpha}}\\ \alpha\ge 2}}p^\alpha\ll x^{\frac{5}{2}}.\end{aligned} \tag{9}$$

Now, combining (7), (8) and (9) we may immediately obtain the fowllowing asymptotic formula:

$$\sum_{n\leq x}(P(n)-p(n))SL(n)=\zeta(3)\cdot x^3\cdot\sum_{i=1}^{k}\frac{b_i}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),$$

where $P(n)$ and $p(n)$ denote the largest and smallest prime divisor of n respectively, $\zeta(s)$ is the Riemann zeta-function, $b_1=\frac{1}{3}$, b_i $(i=2,3,\cdots,k)$ are computable constants.

This completes the proof of Theorem.

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On the Pseudo Smarandache function and its two conjectures

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Abstract For any positive integer n, the famous Pseudo Smarandache function $Z(n)$ is defined as the smallest integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is, $Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\}$, where N denotes the set of all positive integers. The main purpose of this paper is using the elementary method to study the properties of the Pseudo Smarandache function $Z(n)$, and solve two conjectures posed by Kenichiro Kashihara in reference [2].

Keywords Pseudo Smarandache function, conjecture, unbounded.

§1. Introduction and Results

For any positive integer n, the famous Pseudo Smarandache function $Z(n)$ is defined as the smallest positive integer m such that n evenly divides $\sum_{k=1}^{m} k$. That is,

$$Z(n) = \min\left\{m:\ n|\frac{m(m+1)}{2},\ m \in N\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are:

$$Z(1) = 1,\ Z(2) = 3,\ Z(3) = 2,\ Z(4) = 7,\ Z(5) = 4,\ Z(6) = 3,\ Z(7) = 6,\ Z(8) = 15,$$
$$Z(9) = 8, Z(10) = 4,\ Z(11) = 10,\ Z(12) = 8,\ Z(13) = 12,\ Z(14) = 7,\ Z(15) = 5,$$
$$Z(16) = 31,\ Z(17) = 16,\ Z(18) = 8,\ Z(19) = 18,\ Z(20) = 15,\ \cdots\cdots .$$

This function was introduced by David Gorski in reference [3], where he studied the elementary properties of $Z(n)$, and obtained a series interesting results. For example, he proved that if $p \geq 2$ is a prime, then $Z(p) = p - 1$; If $n = 2^k$, then $Z(n) = 2^{k+1} - 1$. The other contents related to the Pseudo Smarandache function can also be found in references [2], [4] and [5]. Especially in reference [2], Kenichiro Kashihara posed two problems as follows: Are the following values bounded or unbounded?

A) $|Z(n+1) - Z(n)|$,

B) $\dfrac{Z(n+1)}{Z(n)}$.

About these two problems, it seems that none had studied it, at least we have not seen related papers before. In this paper, we use the elementary method to study these two problems, and prove that they are unbounded. That is, we shall prove the following conclusion:

Theorem. For any positive number M large enough, there are infinitely positive integers n, such that

$$\frac{Z(n+1)}{Z(n)} > M \text{ and } |Z(n+1) - Z(n)| > M.$$

From this theorem, we know that $|Z(n+1) - Z(n)|$ and $\dfrac{Z(n+1)}{Z(n)}$ are unbounded. This solved two problems posed by Kenichiro Kashihara in reference [2].

§2. Proof of the theorem

In order to complete the proof of the theorem, we need the following important conclusion:

Lemma. Let k and h are any positive integers with $(h,\ k) = 1$, then there are infinitely many primes in the arithmetic progression $nk + h$, where $n = 0,\ 1,\ 2,\ 3,\ \cdots$.

Proof. This is the famous Dirichlet's Theorem, see reference [6].

Now we use this Lemma to complete the proof of our Theorem. In fact for any positive number M, we take positive integer m such that $2^m > M$. Note that $\left(2^{2m+1},\ 2^m + 1\right) = 1$, so from Dirichlet's Theorem we can easily deduce that there are infinitely many primes in the arithmetic progression:

$$2^{2m+1}k + 2^m + 1, \quad \text{where} \quad k = 0,\ 1,\ 2,\ \cdots.$$

Therefore, there must exist a positive integer k_0 such that $2^{2m+1}k_0 + 2^m + 1 = P$ be a prime. For this prime P, from the definition and properties of $Z(n)$ we can deduce that

$$\begin{aligned} &Z(P) = P - 1 = 2^{2m+1}k_0 + 2^m, \\ &Z(P-1) = Z(2^{2m+1}k_0 + 2^m) = Z(2^m(2^{m+1}k_0 + 1)). \end{aligned}$$

Since

$$\sum_{i=1}^{2^{m+1}k_0} i = \frac{2^{m+1}k_0(2^{m+1}k_0 + 1)}{2}$$

and $2^m(2^{m+1}k_0 + 2^m)$ evenly divides $\displaystyle\sum_{i=1}^{2^{m+1}k_0} i$, so we have

$$Z(P-1) \leq 2^{m+1}k_0.$$

Thus

$$\frac{Z(P)}{Z(P-1)} \geqslant \frac{2^{2m+1}k_0 + 2^m}{2^{m+1}k_0} > 2^m > M.$$

So $\dfrac{Z(P)}{Z(P-1)}$ is unbounded.

Similarly, we have

$$\begin{aligned} |Z(P) - Z(P-1)| &\geq |Z(P)| - |Z(P-1)| \\ &\geq 2^{2m+1}k_0 + 2^m - 2^{m+1}k_0 \\ &= 2^{m+1}k_0(2^m - 1) + 2^m > 2^m > M. \end{aligned}$$

So $|Z(P) - Z(P-1)|$ is also unbounded.

Since there are infinitely positive integers m, such that $2^m > M$, so there are infinitely positive integers n, such that $|Z(n+1) - Z(n)|$ and $\dfrac{Z(n+1)}{Z(n)}$ are unbounded.

This completes the proof of the theorem.

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Diophantine equations and their positive integer solutions

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Abstract In his book "Comments and Topics on Smarandache Notions and Problems", Kenichiro Kashihara asked us to find all positive integer solutions of the Diophantine equations $\left(a^b \cdot b^a\right)^{\frac{2}{a+b}} = c$ or $\left(a^b \cdot b^c \cdot c^a\right)^{\frac{3}{a+b+c}} = d$. He also said, "The question I would like to ask is whether there exists positive integer solutions where $a \neq b$ for either of the equations. If such non-trivial solutions exist, find a general method for generating them". In this paper, we using the elementary method to study these two problems, and prove that the first equation has no non-trivial solutions. The second equation has infinite non-trivial solutions. This solved the problems proposed by Kenichiro Kashihara in his book.

Keywords Diophantine equation, positive integer solutions, elementary method.

§1. Introduction and results

It is a very important content in number theory that to find all integer (or positive integer) solutions for some Diophantine equations. For example, the famous Fermat's last theorem is that the Diophantine equation

$$x^n + y^n = z^n$$

has no positive integer solutions for all integer $n \geq 3$.

In his book "Comments and Topics on Smarandache Notions and Problems", Kenichiro Kashihara asked us to find all positive integer solutions of the Diophantine equations

$$\left(a^b \cdot b^a\right)^{\frac{2}{a+b}} = c \tag{1}$$

or

$$\left(a^b \cdot b^c \cdot c^a\right)^{\frac{3}{a+b+c}} = d. \tag{2}$$

He also said: "The question I would like to ask is whether there exists positive integer solutions where $a \neq b$ for either of the equations. If such non-trivial solutions exist, find a general method for generating them". In this paper, we using the elementary method to study these two problems, and prove that the Diophantine equation (1) has no non-trivial positive integer solutions, and the Diophantine equation (2) has infinite non-trivial positive integer solutions. That is, we shall prove the following conclusions:

Theorem 1. All positive integers a, b and c satisfying the equation

$$\left(a^b \cdot b^a\right)^{\frac{2}{a+b}} = c$$

if and only if $a = b = s$ and $c = s^2$, where s is any positive integer.

Theorem 2. If $(a,\ b,\ c) = t$, $a = a_1 t$, $b = b_1 t$, $c = c_1 t$ and $(a_1,\ b_1) = (a_1,\ c_1) = (c_1,\ b_1) = 1$, then $a,\ b,\ c$ and d does not satisfy the equation (2);

If $(a,\ b,\ c,\ d) = (t,\ t,\ t,\ t^3)$, $(t,\ t,\ 4t,\ 2t^3)$, $(4t,\ t,\ t,\ 2t^3)$ or $(t,\ 4t,\ t,\ 2t^3)$, then a, b, c and d satisfying the equation (2), where t is any positive integer.

It is clear that our Theorem 1 and Theorem 2 solved two problems proposed by Kenichiro Kashihara in reference [2]. Whether there exists any other positive integer solutions for the equation (2) is an open problem.

§2. Proof of the theorem

In this section, we shall prove our Theorems directly. First we prove Theorem 1. For any positive integer a, b and c, if they satisfy the Diophantine equation (1), then we have

$$\left(a^b \cdot b^a\right)^{\frac{2}{a+b}} = c. \tag{3}$$

Let $(a,\ b) = d$ denotes the Greatest Common Divisor of a and b, and $a = d \cdot a_1$, $b = d \cdot b_1$. Then $(a_1,\ b_1) = 1$, and from equation (3) we have

$$a_1^{\frac{2b_1}{a_1+b_1}} \cdot b_1^{\frac{2a_1}{a_1+b_1}} \cdot d^2 = c. \tag{4}$$

In equation (4), since $(a_1,\ b_1) = 1$, c and d are positive integers, so $a_1^{\frac{2b_1}{a_1+b_1}}$ and $b_1^{\frac{2a_1}{a_1+b_1}}$ both must be positive integers. If $a_1 = b_1$, then (4) become $a_1 \cdot b_1 \cdot d^2 = a^2 = c$. So all positive integers $a = b$ and $c = a^2$ must be the solutions of the Diophantine equation (1). If $a_1 \neq b_1$, then without loss of generality we assume that $a_1 > b_1$. Now we prove that $a_1^{\frac{2b_1}{a_1+b_1}}$ can not be a positive integer. In fact for any prime divisor p of a_1 with $p^\alpha \mid a_1$ and $p^{\alpha+1} \nmid a_1$, if $a_1^{\frac{2b_1}{a_1+b_1}}$ be an integer, then $p^{\frac{2\alpha b_1}{a_1+b_1}}$ also be an integer, and so $\dfrac{2\alpha b_1}{a_1+b_1}$ must be an integer. Note that $(a_1,\ b_1) = 1$, $(a_1+b_1,\ b_1) = 1$ and $\dfrac{2\alpha b_1}{a_1+b_1}$ be an integer, we can deduce that

$$a_1 + b_1 \mid 2\alpha. \tag{5}$$

It is clear that $p^\alpha \mid a_1$, so the formula (5) implies that

$$2\alpha \geq a_1 + b_1 > a_1 = p^\alpha \cdot a_2, \tag{6}$$

where a_2 be a positive integer.

Since p be a prime and α be a positive integer, we must have $p^\alpha \cdot a_2 \geq p^\alpha \geq 2^\alpha \geq 2\alpha$. So the inequality (6) is impossible. Therefore, all positive integers a, b and c satisfying the Diophantine equation (1) if and only if $a = b$ and $c = a^2$. This proves Theorem 1.

Now we prove Theorem 2. If positive integers a, b, c and d satisfying the Diophantine equation (2). Let $(a,\ b,\ c) = t$, $a = t \cdot a_1$, $b = t \cdot b_1$ and $c = t \cdot c_1$, then $(a_1,\ b_1,\ c_1) = 1$, so from the equation (2) we have the identity

$$a_1^{\frac{3b_1}{a_1+b_1+c_1}} \cdot b_1^{\frac{3c_1}{a_1+b_1+c_1}} \cdot c_1^{\frac{3a_1}{a_1+b_1+c_1}} \cdot t^3 = d. \tag{7}$$

If $a_1 = b_1 = c_1$ in the above formula, then note that $(a_1,\ b_1,\ c_1) = 1$, we have $a_1 = b_1 = c_1 = 1$ and $d = t^3$. So all positive integers $a = b = c$ and $d = a^3$ are the solutions of the Diophantine equation (2).

If $a_1 = b_1 < c_1$, then we can change the equation (7) into

$$a_1^{\frac{3(a_1+c_1)}{2a_1+c_1}} \cdot c_1^{\frac{3a_1}{2a_1+c_1}} \cdot t^3 = d. \tag{8}$$

Note that $(a_1,\ c_1) = 1$, so $a_1^{\frac{3(a_1+c_1)}{2a_1+c_1}}$ and $c_1^{\frac{3a_1}{2a_1+c_1}}$ must be integers. Therefore, for any prime divisor p of c_1 with $p^\alpha \mid c_1$ and $p^{\alpha+1} \nmid c_1$, $\dfrac{3a_1\alpha}{2a_1+c_1}$ must be an integer, or $(2a_1+c_1) \mid 3a_1\alpha$. Since $(2a_1+c_1,\ c_1) = 1$, we deduce that $(2a_1+c_1) \mid 3\alpha$ and $3\alpha = k(2a_1+c_1)$. So

$$3\alpha \geq 2a_1 + c_1 = 2a_1 + p^\alpha c_2 \geq 2a_1 + p^\alpha$$

or

$$3\alpha - 2a_1 \geq p^\alpha.$$

This inequality holds if and only if $a_1 = 1$, $p = 2 = \alpha$. This time, $a = b = t$, $c = 4t$ and $d = 2t^3$. So for any positive integer t, $a = b = t$, $c = 4t$ and $d = 2t^3$ are the solution of the equation (2).

If $a_1 = b_1 > c_1$, then for any prime divisor p of a_1 with $p^\beta \mid a_1$ and $p^{\beta+1} \nmid a_1$, $a_1^{\frac{3(a_1+c_1)}{2a_1+c_1}}$ must be an integer, or $(2a_1+c_1) \mid 3(a_1+c_1)\beta$. Since $(2a_1+c_1,\ a_1+c_1) = 1$, we deduce that $(2a_1+c_1) \mid 3\beta$ or $3\beta = k(2a_1+c_1)$. So $3\beta \geq 2a_1 + c_1$, $3\beta - 1 \geq 2p^\beta$. This is not possible, since $3\beta - 1 < 2p^\beta$ for all positive integer β.

If $a_1 > b_1 > c_1$, first we prove that if $(a_1,\ b_1) = (a_1,\ c_1) = (c_1,\ b_1) = 1$, then (7) is not possible. In fact if $(a_1,\ b_1) = (a_1,\ c_1) = (c_1,\ b_1) = 1$, then

$$a_1^{\frac{3b_1}{a_1+b_1+c_1}},\quad b_1^{\frac{3c_1}{a_1+b_1+c_1}},\quad \text{and}\quad c_1^{\frac{3a_1}{a_1+b_1+c_1}}$$

all must be positive integers. Let $(a_1,\ b_1+c_1) = u$, $(b_1,\ a_1+c_1) = v$, $(c_1,\ a_1+b_1) = w$. Note that $(a_1,\ b_1,\ c_1) = 1$, so it is clear that $(u,\ v) = (v,\ w) = (w,\ u) = 1$. If $c_1 = 1$ or 2, then $b_1^{\frac{3c_1}{a_1+b_1+c_1}}$ is not a positive integer. So without loss of generality, we can assume that $c_1 \geq 3$. If $u \leq \sqrt{a_1}$, then $c_1^{\frac{3a_1}{a_1+b_1+c_1}}$ is not an integer. Otherwise, for any prime divisor p with $p^\alpha \mid c_1$ and $p^{\alpha+1} \nmid c_1$, $\dfrac{3a_1\alpha}{a_1+b_1+c_1}$ must be an integer, so $3a_1\alpha = t(a_1+b_1+c_1)$. or $\dfrac{3a_1\alpha}{u} = t \cdot \dfrac{(a_1+b_1+c_1)}{u}$, note that $\left(\dfrac{a_1}{u},\ \dfrac{a_1+b_1+c_1}{u}\right) = 1$, so that $\frac{a_1}{u} \mid t$. Therefore, we have

$$3\alpha = t_2 \cdot \frac{a_1}{u^2} \cdot (a_1+b_1+c_1) \geq a_1+b_1+c_1 \geq 3c_1+3 \geq 3p^\alpha + 3.$$

This is not possible. Similarly, if $v \leq \sqrt{b_1}$ or $w \leq \sqrt{c_1}$, then $a_1^{\frac{3b_1}{a_1+b_1+c_1}}$ or $b_1^{\frac{3c_1}{a_1+b_1+c_1}}$ is not an integer. So we can assume that $u > \sqrt{a_1}$, $v > \sqrt{b_1}$ and $w > \sqrt{c_1}$.

If $\sqrt{a_1} < u \leq \dfrac{a_1}{2}$, then note that $u \mid a_1+b_1+c_1$, $v \mid a_1+b_1+c_1$ and $w \mid a_1+b_1+c_1$, so $c_1^{\frac{3a_1}{a_1+b_1+c_1}}$ is not an integer. In fact this time, for any prime divisor p with $p^\alpha \mid c_1$ and $p^{\alpha+1} \dagger c_1$, $\dfrac{3a_1\alpha}{a_1+b_1+c_1}$ must be an integer, so $3a_1\alpha = t(a_1+b_1+c_1)$ or

$$3\alpha = t_2 \cdot \frac{a_1}{u} \cdot v \cdot w \cdot x \geq 2v \cdot w > 2c_1 \geq 2p^\alpha.$$

This is not possible.

Similarly, if $\sqrt{c_1} < w \leq \dfrac{c_1}{2}$, then $b_1^{\frac{3c_1}{a_1+b_1+c_1}}$ is not an integer.

If $w > \dfrac{c_1}{2}$ and $u > \dfrac{a_1}{2}$, then note that $u \mid a_1$ and $w \mid c_1$, so this time, we have $a_1 = u$ and $c_1 = w$. From $u \mid a_1+b_1+c_1$, $w \mid a_1+b_1+c_1$ and $(u,\ w) = 1$ we have $u \cdot w \mid a_1+b_1+c_1$. Therefore, $a_1+b_1+c_1 \geq u \cdot w = a_1 \cdot c_1 \geq 3a_1 > a_1+b_1+c_1$. This is not possible.

Therefore, if $a_1 > b_1 > c_1$ and $(a_1,\ b_1) = (a_1,\ c_1) = (c_1,\ b_1) = 1$, then (7) is not possible.

From the above we know that if $(a,\ b,\ c) = t$, $a = a_1t$, $b = b_1t$, $c = c_1t$ and $(a_1,\ b_1) = (a_1,\ c_1) = (c_1,\ b_1) = 1$, then $a,\ b,\ c$ and d does not satisfy the equation (2). If $(a,\ b,\ c,\ d) = (t,\ t,\ t,\ t^3)$, $(t,\ t,\ 4t,\ 2t^3)$, $(4t,\ t,\ t,\ 2t^3)$ or $(t,\ 4t,\ t,\ 2t^3)$, then $a,\ b,\ c$ and d satisfying the equation (2).

This completes the proof of Theorem 2.

Whether there exists any other positive integer solutions for the equation (2) is an open problem.

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On the Pseudo-Smarandache function

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Abstract The main purpose of this paper is using the elementary method to study the properties of the Pseudo-Smarandache function $Z(n)$, and proved the following two conclusions: The equation $Z(n) = Z(n+1)$ has no positive integer solutions; For any given positive integer M, there exists an integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than M.

Keywords Pseudo-Smarandache function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the Pseudo-Smarandache function $Z(n)$ is defined as the smallest positive integer m such that $[1 + 2 + 3 + \cdots + m]$ is divisible by n. That is,

$$Z(n) = \min\left\{m : \ m \in N : \ n \mid \frac{m(m+1)}{2}\right\},$$

where N denotes the set of all positive integers. For example, the first few values of $Z(n)$ are: $Z(1) = 1,\ Z(2) = 3,\ Z(3) = 2,\ Z(4) = 7,\ Z(5) = 4,\ Z(6) = 3,\ Z(7) = 6,\ Z(8) = 15,\ Z(9) = 8, Z(10) = 4,\ Z(11) = 10,\ Z(12) = 8,\ Z(13) = 12,\ Z(14) = 7,\ Z(15) = 5, \cdots\cdots$.

In reference [1], Kenichiro Kashihara had studied the elementary properties of $Z(n)$, and proved some interesting conclusions. Some of them as follows:

For any prime $p \geq 3$, $Z(p) = p - 1$;

For any prime $p \geq 3$ and any $k \in N, Z(p^k) = p^k - 1$;

For any $k \in N, Z(2^k) = 2^{k+1} - 1$;

If n is not the form 2^k for some integer $k > 0$, then $Z(n) < n$.

On the other hand, Kenichiro Kashihara proposed some problems related to the Pseudo-Smarandache function $Z(n)$, two of them as following:

(A) Show that the equation $Z(n) = Z(n + 1)$ has no solutions.

(B) Show that for any given positive number r, there exists an integer s such that the absolute value of $Z(s) - Z(s + 1)$ is greater than r.

For these two problems, Kenichiro Kashihara commented that I am not able to solve them, but I guess they are true. I checked it for $1 \leq n \leq 60$.

In this paper, we using the elementary method to study these two problems, and solved them completely. That is, we shall prove the following:

Theorem 1. The equation $Z(n) = Z(n+1)$ has no positive integer solutions.

Theorem 2. For any given positive integer M, there exists a positive integer s such that

$$|Z(s) - Z(s+1)| > M.$$

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. If there exists some positive integer n such that the equation $Z(n) = Z(n+1)$. Let $Z(n) = Z(n+1) = m$, then from the definition of $Z(n)$ we can deduce that

$$n \mid \frac{m(m+1)}{2}, \quad n+1 \mid \frac{m(m+1)}{2}.$$

Since $(n,\ n+1) = 1$, we also have

$$n(n+1) \mid \frac{m(m+1)}{2} \quad \text{and} \quad \frac{n(n+1)}{2} \mid \frac{m(m+1)}{2}.$$

Therefore,

$$n < m. \tag{1}$$

On the other hand, since one of n and $n+1$ is an odd number, if n is an odd number, then $Z(n) = m \leq n - 1 < n$; If $n+1$ is an odd number, then $Z(n+1) = m \leq n$. In any cases, we have

$$m \leq n. \tag{2}$$

Combining (1) and (2) we have $n < m \leq n$, it is not possible. This proves Theorem 1.

Now we prove Theorem 2. For any positive integer M, we taking positive integer α such that $s = 2^\alpha > M + 1$. This time we have

$$Z(s) = Z(2^\alpha) = 2^{\alpha+1} - 1.$$

Since $s + 1$ is an odd number, so we have

$$Z(s+1) \leq s = 2^\alpha.$$

Therefore, we have

$$|Z(s) - Z(s+1)| \geq (2^{\alpha+1} - 1) - 2^\alpha = 2^\alpha - 1 > M + 1 - 1 = M.$$

So there exists a positive integer s such that the absolute value of $Z(s) - Z(s+1)$ is greater than M. This completes the proof of Theorem 2.

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On a problem related to function $S(n)$

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Abstract For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. The main purpose of this paper is using the elementary method to study the number of all positive integer n such that $\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)}$ is a positive integer.

Keywords F.Smarandache function $S(n)$, related problem, positive integer solution.

§1. Introduction and Result

For any positive integer n, the famous F.Smarandache function $S(n)$ is defined as the smallest positive integer m such that n divides $m!$. That is, $S(n) = \min\{m :\ m \in N,\ n|m!\}$, where N denotes the set of all positive integers. From the definition of $S(n)$, it is easy to see that if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ be the factorization of n into prime powers, then we have

$$S(n) = \max_{1\le i\le k}\{S(p_i^{\alpha_i})\}.$$

It is clear that from this properties we can get the value of $S(n)$, the first few values of $S(n)$ are $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, $S(9) = 6$, $S(10) = 5$, $\cdots\cdots$. About the arithmetical properties of $S(n)$, some authors had studied it, and obtained some interesting results. For example, Farris Mark and Mitchell Patrick [1] studied the bound of $S(n)$, and got the upper and lower bound estimates for $S(p^\alpha)$. They proved that

$$(p-1)\alpha + 1 \le S(p^\alpha) \le (p-1)[\alpha + 1 + \log_p \alpha] + 1.$$

Lu Yaming [2] studied the solutions of an equation involving the F.Smarandache function $S(n)$, and proved that for any positive integer $k \ge 2$, the equation

$$S(m_1 + m_2 + \cdots + m_k) = S(m_1) + S(m_2) + \cdots + S(m_k)$$

has infinite positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Jozsef Sandor [3] proved that for any positive integer $k \ge 2$, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ satisfying the inequality:

$$S(m_1 + m_2 + \cdots + m_k) > S(m_1) + S(m_2) + \cdots + S(m_k).$$

Also, there exist infinite group positive integers $(m_1,\ m_2,\ \cdots,\ m_k)$ such that

$$S(m_1+m_2+\cdots+m_k)<S(m_1)+S(m_2)+\cdots+S(m_k).$$

In [4], Fu Jing proved a more general conclusion. That is, if the positive integer k and m satisfying one of the following conditions:

(a) $k>2$ and $m\geq 1$ are odd numbers.

(b) $k\geq 5$ is odd, $m\geq 2$ is even.

(c) Any even number $k\geq 4$ and any positive integer m;

then the equation

$$m\cdot S(m_1+m_2+\cdots+m_k)=S(m_1)+S(m_2)+\cdots+S(m_k)$$

have infinite group positive integer solutions $(m_1,\ m_2,\ \cdots,\ m_k)$.

Xu Zhefeng [5] studied the value distribution properties of $S(n)$, and obtained a deeply result. That is, he proved the following Theorem:

Let $P(n)$ denotes the largest prime factor of n. Then for any real number $x>1$, we have the asymptotic formula

$$\sum_{n\leq x}\left(S(n)-P(n)\right)^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function.

On the other hand, in the manuscript "Problems lists for collective book on Smarandache notions", Kenichiro Kashihara proposed the following problem: Find all positive integer $n\in N$ such that

$$\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)} \tag{1}$$

is an integer. About this problem, it seems that none had studied it yet, at least we have not seen related papers before. In this paper, we using the elementary method to study this problem, and solved it completely. We shall prove the following conclusion:

Theorem. For any positive integer n, the formula

$$\frac{S(2)\cdot S(4)\cdot S(6)\cdot\ \cdots\ S(2n)}{S(1)\cdot S(3)\cdot S(5)\cdot\ \cdots\ S(2n-1)}$$

is an integer if and only if $n=1$.

§2. Proof of the theorem

In this section, we shall use the elementary method to complete the proof of the theorem. First we need the following two simple lemmas.

Lemma 1. For any positive integer $n\geq 5$, there exists at least one prime P such that $P\in(n,\ 2n-1]$.

Proof. See Theorem 5.7.1 of reference [6].

Lemma 2. Let p be a prime. Then for any positive integer k, we have the estimate $S(p^k) \leq kp$. If $k \leq p$, then $S(p^k) = kp$.

Proof. See reference [1].

Now we use these two lemmas to complete the proof of our theorem. It is clear that if $n = 1$, then

$$\frac{S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n)}{S(1) \cdot S(3) \cdot S(5) \cdot \ \cdots \ S(2n-1)}$$

is an integer. In fact, this time we have

$$\frac{S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n)}{S(1) \cdot S(3) \cdot S(5) \cdot \ \cdots \ S(2n-1)} = \frac{S(2)}{S(1)} = 1.$$

If $n = 2$, we have $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, obviously formula (1) is not an integer.

Similarly, if $n = 3$ and 4, we have $S(1) = 1$, $S(2) = 2$, $S(3) = 3$, $S(4) = 4$, $S(5) = 5$, $S(6) = 3$, $S(7) = 7$, $S(8) = 4$, obviously (1) is not also an integer.

Now we assume that $n > 4$. From Lemma 1 we know that there exists at least one prime $P \in (n,\ 2n-1]$ such that $S(P) = P$. For this prime P, we have $P \mid S(1) \cdot S(3) \cdot S(5) \cdot \cdots \ S(2n-1)$. But we can prove that

$$P \ \dagger \ S(2) \cdot S(4) \cdot S(6) \cdot \ \cdots \ S(2n).$$

Otherwise, there exists an integer k with $1 \leq k \leq n$ such that $P \mid S(2k)$. Let $S(2k) = \alpha P$. If $\alpha = 1$, then $S(2k) = P$ and $P \mid 2k$. So $P \mid k$. Therefore, $k \geq P \geq n+1$. This is a contradiction with $1 \leq k \leq n$. If $\alpha \geq 2$, then from Lemma 2 we have $2k \geq S(2k) = \alpha P \geq 2P \geq 2(n+1)$, or $k \geq P \geq n+1$, contradict with $1 \leq k \leq n$.

This completes the proof of our theorem.

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On γ-s-closed spaces

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Abstract The concept of generalized open sets in generalized topological spaces was introduced by A. Csaszar [7,8]. In this paper, we introduce a class of topological spaces called γ-s-closed spaces by utilizing the γ^*-semi-closure. In [16] Maio and Noiri showed that cd-compactness due to Carnahan [3], weak RS-compactness due to Hong [11] and s-closedness are all equivalent [16]. It is shown that the concept of γ-s-closed spaces generalized s-closed spaces [17]. It is interesting to note that γ-s-closedness is the generalization of γ_0-compactness defined and investigated in [2].

Keywords γ-closed (open), γ-closure, γ^*-semi-open (closed), γ^*-semi-closure, γ^*-semi-interior, γ-regular, γ-regular-open (closed), γ^*-semi-regular, γ-s-closed.

§1. Introduction

A. Csaszar [7,8] defined generalized open sets in generalized topological spaces. In 1979, S. Kasahara [12] defined an operation α on topological spaces. In 1992 (1993), B. Ahmad and F.U. Rehman [1,19] introduced the notions of γ-interior, γ-boundary and γ-exterior points in topological spaces. They also studied properties and characterizations of (γ, β)-continuous mappings introduced by H. Ogata [18].

In 2005, A. Guldurdek and O.B. Ozbakir [10] defined and discussed γ-semi-open sets using γ-open sets in topological spaces which are different from the notions of γ-open sets introduced and studied by H.Ogata [17] in 1991. In 2006, S.Hussain, B. Ahmad and T. Noiri [6] defined and discussed γ^*-semi-open sets, γ^*-semi-closed sets, γ^*-semi-closure sets, γ^*-semi-interior sets in a space X in the sense of H. Ogata [18].

In 1987, G. Di Maio and T. Noiri [16] introduced the notion of s-closed spaces. It was shown that cd-compactness due to Carnahan [3], weak RS-compactness due to Hong [11] and s-closedness are all equivalent [16].

In this paper, we introduce a class of topological spaces called γ-s-closed spaces by utilizing the γ^*-semi-closure. It is shown that the concept of γ-s-closed spaces generalized s-closed spaces [17]. It is interesting to note that γ-s-closedness is the generalization of γ_0-compactness defined and investigated in [2].

Hereafter, we shall write spaces in place of topological spaces on which no separation axiom is assumed explicitly. We recall some definitions and results used in this paper to make it self-contained.

§2. Preliminaries

Definition 2.1.[12] Let (X, τ) be a space. An operation $\gamma : \tau \to P(X)$ is a function from τ to the power set of X such that $V \subseteq V^\gamma$, for each $V \in \tau$, where V^γ denotes the value of γ at V. The operations defined by $\gamma(\mathrm{G}) = \mathrm{G}$, $\gamma(\mathrm{G}) = \mathrm{cl}(\mathrm{G})$ and $\gamma(\mathrm{G}) = \mathrm{int\ cl}(\mathrm{G})$ are examples of operation γ.

Definition 2.2.[18] Let A be a subset of a space X. A point $x \in A$ is said to be a γ-interior point of A if there exists an open nbd N of x such that $N^\gamma \subseteq A$ and we denote the set of all such points by $\mathrm{int}_\gamma(A)$. Thus

$$\mathrm{int}_\gamma(\mathrm{A}) = \{\mathrm{x} \in \mathrm{A} : \mathrm{x} \in \mathrm{N} \in \tau \ \text{ and } \ \mathrm{N}^\gamma \subseteq \mathrm{A}\} \subseteq \mathrm{A}.$$

Note that A is γ-open [18] iff $\mathrm{A} = \mathrm{int}_\gamma(\mathrm{A})$.

A set A is called γ- closed [18] iff X-A is γ-open.

Definition 2.3.[12] A point $x \in X$ is called a γ-closure point of $A \subseteq X$, if $U^\gamma \cap A \neq \phi$, for each open nbd U of x. The set of all γ-closure points of A is called γ-closure of A and is denoted by $\mathrm{cl}_\gamma(\mathrm{A})$. A subset A of X is called γ-closed, if $\mathrm{cl}_\gamma(A) \subseteq A$. Note that $\mathrm{cl}_\gamma(\mathrm{A})$ is contained in every γ-closed superset of A.

Definition 2.4.[19] An operation γ on τ is said to be regular, if for any open nbds U, V of $x \in X$, there exists an open nbd W of x such that $U^\gamma \cap V^\gamma \supseteq W^\gamma$.

Definition 2.5.[18] An operation γ on τ is said to be open, if for every nbd U of $x \in X$, there exists a γ-open set B such that $x \in B$ and $U^\gamma \supseteq B$.

Definition 2.6.[6] A subset A of a space (X, τ) is said to be a γ^*-semi-open set, if there exists a γ-open set O such that $O \subseteq A \subseteq cl_\gamma(O)$. The set of all γ^*-semi-open sets is denoted by $SO_{\gamma^*}(X)$. A is γ^*-semi-closed iff $X - A$ is γ^*-semi-open in X. Note that A is γ^*-semi-closed if and only if $\mathrm{int}_\gamma(cl_\gamma(A)) \subseteq A$.

We denote $\Gamma(X)$, the set of all monotone operators.

Definition 2.7.[6] Let A be a subset of a space X and $\gamma \in \Gamma(X)$. The intersection of all γ^*-semi-closed sets containing A is called γ^*-semi-closure of A and is denoted by $scl_{\gamma^*}(A)$. Note that A is γ^*-semi-closed if and only if $scl_{\gamma^*}(A) = A$.

Definition 2.8.[6] Let A be a subset of a space X and $\gamma \in \Gamma(X)$. The union of γ^*-semi-open subsets of A is called γ^*-semi-interior of A and is denoted by $\mathrm{sint}_{\gamma^*}(\mathrm{A})$.

§3. γ^*-Semi-Regular Sets

Definition 3.1. A subset A of a space X is called γ-regular-open, if $A = \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{A}))$. The set of γ-regular open sets is denoted by $RO_\gamma(X, \tau)$.

Note that $RO_\gamma(X, \tau) \subseteq \tau_\gamma \subseteq \tau$.

Definition 3.2. A set A is γ-regular closed, denoted by $RC_\gamma(X,\tau)$, if one of the following conditions holds:

$$(i)A = cl_\gamma(\text{int}_\gamma(\text{A})).$$

$$(ii)X - A \in RO_\gamma(X,\tau).$$

Clearly A is γ-regular open if and only if $X - A$ is γ-regular closed.

Definition 3.3.[6] A subset A of a space X is said to be γ^*-semi-regular, if it is both γ^*-semi-open and γ^*-semi-closed.The class of all γ^*-semi-regular sets of X is denoted by $SR_{\gamma^*}(A)$.

Note that if γ is regular operation, then the union of γ^*-semi-regular sets is γ^*-semi-regular.

Definition 3.4. A subset A of X is said to be γ^*-semi-regular-open if $A = \text{sint}_{\gamma^*}(\text{scl}_{\gamma^*}(\text{A}))$.

Lemma 3.5. If A is a subset of a space X, then $\text{int}_\gamma(\text{cl}_\gamma(\text{A})) \subseteq \text{scl}_{\gamma^*}(\text{A})$.

Proof. Let $x \in int_\gamma(cl_\gamma(A))$. Let G be a γ^*-semi-open set of Xcontaining x. Then $U \subseteq G \subseteq cl_\gamma(U)$, for some γ-open set U in X.

Since $x \in G \subseteq cl_\gamma(U)$ and $x \in \text{int}_\gamma(\text{cl}_\gamma(\text{A}))$, we have

$$\phi \neq \text{int}_\gamma(\text{cl}_\gamma(\text{A})) \cap \text{U} \subseteq \text{cl}_\gamma(\text{A}) \cap \text{U} \subseteq \text{cl}_\gamma(\text{A} \cap \text{U}). \quad (\text{byLemma2(3)[19]})$$

Therefore, we have $A \cap U \neq \phi$ and hence $A \cap G \neq \phi$. This shows that $x \in scl_{\gamma^*}(A)$.

Hence the proof.

We use Lemma 3.5 and prove:

Proposition 3.6. For any subset A of a space X, the following are equivalent:

(1)$A \in SR_{\gamma^*}(X)$.

(2)$A = \text{sint}_{\gamma^*}(\text{scl}_{\gamma^*}(\text{A}))$.

(3) There exists a γ-semi-regular-open set U of X such that $U \subseteq A \subseteq cl_\gamma(U)$, where γ is an open operation.

Proof. (1)⇒(2). If $A \in SR_{\gamma^*}(X)$, then $\text{sint}_{\gamma^*}(\text{scl}_{\gamma^*(\text{A})}) = \text{sin}_{\gamma^*}(\text{A}) = \text{A}$.

(2)⇒(3). Suppose that $A = \text{sint}_{\gamma^*}(\text{scl}_{\gamma^*}(\text{A}))$. By Lemma 3.5, for any subset A of X, $\text{int}_\gamma(\text{cl}_\gamma(\text{A})) \subseteq \text{scl}_{\gamma^*}(\text{A})$ implies

$$\text{int}_\gamma(\text{cl}_\gamma(\text{A})) \subseteq \text{sint}_{\gamma^*}(\text{scl}_{\gamma^*}(\text{A})) = \text{A}.$$

Since $A \in SO_{\gamma^*}(X)$, we have $A \subseteq cl_\gamma(\text{int}_\gamma(\text{A}))$. Therefore, we obtain

$$\text{int}_\gamma(\text{cl}_\gamma(\text{A})) \subseteq \text{A} \subseteq \text{cl}_\gamma(\text{int}_\gamma(\text{A})) \subseteq \text{cl}_\gamma(\text{int}_\gamma(\text{cl}_\gamma(\text{A}))),$$

where $\text{int}_\gamma(\text{cl}_\gamma(\text{A}))$ is γ-regular open, since $\text{int}_\gamma(\text{cl}_\gamma(\text{int}_\gamma(\text{cl}_\gamma(\text{A}))))) = \text{int}_\gamma(\text{cl}_\gamma(\text{A}))$.

(3)⇒(1). It is obvious that $A \in SO_{\gamma^*}(X)$. Clearly from (3) and Theorem 3.6[18], we have $\text{int}_\gamma(\text{cl}_\gamma(\text{A})) = \text{int}_\gamma(\text{cl}_\gamma(\text{U})) = \text{U} \subseteq \text{A}$ and hence A is γ^*-semi-closed.

Thus, we obtain $A \in SR_{\gamma^*}(X)$. This completes the proof.

Proposition 3.7. If $A \in SO_{\gamma^*}(X)$, then $scl_{\gamma^*}(A) \in SR_{\gamma^*}(X)$.

Proof. Since $scl_{\gamma^*}(A)$ is γ^*-semi-closed, we show that $scl_{\gamma^*}(A) \in SO_{\gamma^*}(X)$. Since $A \in SO_{\gamma^*}(X)$, then for γ-open set U of X, $U \subseteq A \subseteq cl_\gamma U$.

Therefore we have

$$U \subseteq scl_{\gamma^*}(U) \subseteq scl_{\gamma^*}(A) \subseteq scl_{\gamma^*}(cl_\gamma(U)) = cl_\gamma(U)$$

or

$$U \subseteq scl_{\gamma^*}(A) \subseteq cl_\gamma(U)$$

and hence $scl_{\gamma^*}(A) \in SO_{\gamma^*}(X)$. Hence the proof.

Definition 3.8. A point $x \in X$ is said to be a γ-semi-θ-adherent point of a subset A of X if $scl_{\gamma^*}(U) \cap A \neq \phi$, for every $U \in SO_{\gamma^*}(X)$. The set of all γ-semi-θ-adherent points of A is called the γ-semi-θ-closure of A and is denoted by $s_\gamma cl_\theta(A)$.

A subset A is called γ-semi-θ-closed if $s_\gamma cl_\theta(A) = A$.

Proposition 3.9. Let A be a subset of a space X. Then we have

(1) If $A \in SO_{\gamma^*}(X)$, then $scl_{\gamma^*}(A) = s_\gamma cl_\theta(A)$.

(2) If $A \in SR_{\gamma^*}(X)$, then A is γ-semi-θ-closed.

Proof. (1) Clearly $scl_{\gamma^*}(A) \subseteq s_\gamma cl_\theta(A)$. Suppose that $x \notin scl_{\gamma^*}(A)$. Then, for some γ^*-semi-open set U, $A \cap U = \phi$ and hence $A \cap scl_{\gamma^*}(U) = \phi$, since $A \in SO_{\gamma^*}(X)$. This shows that $x \notin s_\gamma cl_\theta(A)$. Therefore $scl_{\gamma^*}(A) = s_\gamma cl_\theta(A)$.

(2) This follows from (1). Hence the proof.

Lemma 3.10. If B is γ-open set in a space X, then $scl_{\gamma^*}(B) = \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B}))$.

Proof. From Lemma 3.5, we have $\mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B})) \subseteq \mathrm{scl}_{\gamma^*}(\mathrm{B})$, for any subset B of X.

Now we show that $scl_{\gamma^*}(B) \subseteq \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B}))$. For this let $x \notin \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B}))$. Then

$$x \in cl_\gamma(\mathrm{int}_\gamma(\mathrm{X} - \mathrm{B})) \in \mathrm{SO}_{\gamma^*}(\mathrm{X}).$$

Since B is γ-open, we have $B \subseteq \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B}))$ and $B \cap cl_\gamma(\mathrm{int}_\gamma(\mathrm{X} - \mathrm{B})) = \phi$. This shows that $x \notin scl_{\gamma^*}(B)$. Therefore, we obtain $scl_{\gamma^*}(B) = \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{B}))$.

Hence the proof.

Definition 3.11. A space X is γ-extremally disconnected space, if $cl_\gamma(U)$ is γ-open set, for every γ-open set U in X.

Lemma 3.12. A subset A of a space X is γ^*-semi-open if and only if $cl_\gamma(A) = cl_\gamma(\mathrm{int}_\gamma(\mathrm{A}))$, where γ is an open operation.

Proof. Suppose A is γ^*-semi-open. Then we have $A \subseteq cl_\gamma(\mathrm{int}_\gamma(\mathrm{A}))$ and since γ is an open operation, so $cl_\gamma(A) \subseteq cl_\gamma(\mathrm{int}_\gamma(\mathrm{A}))$[19].

On the other hand, we have $\mathrm{int}_\gamma(\mathrm{A}) \subseteq \mathrm{A}$ and hence $cl_\gamma(\mathrm{int}_\gamma(\mathrm{A})) \subseteq \mathrm{cl}_\gamma(\mathrm{A})$. Consequently, we have $cl_\gamma(A) = cl_\gamma(\mathrm{int}_\gamma(\mathrm{A}))$.

Conversely, by the supposition, we have

$$\mathrm{int}_\gamma(\mathrm{A}) \subseteq \mathrm{A} \subseteq \mathrm{cl}_\gamma(\mathrm{A}) = \mathrm{cl}_\gamma(\mathrm{int}_\gamma(\mathrm{A})).$$

Hence A is γ^*-semi-open.

Hence the proof.

Proposition 3.13. A space X is γ-extremally disconnected if and only if $cl_\gamma(U) = scl_{\gamma^*}(U)$, for every $U \subseteq SO_{\gamma^*}(X)$, where γ is a regular and open operation.

Proof. Necessity. In general $scl_{\gamma^*}(U) \subseteq cl_\gamma(U)$, for every subset U of X. We show that $cl_\gamma(U) \subseteq scl_{\gamma^*}(U)$, for each $U \in SO_{\gamma^*}(X)$. Let $\phi \neq scl_{\gamma^*}(U) \in SO_{\gamma^*}(X)$ and let $x \notin scl_{\gamma^*}(U)$, then there exists $V \in SO_{\gamma^*}(X)$ such that $x \in V$ and $V \cap U = \phi$ and hence $\mathrm{int}_\gamma(\mathrm{V}) \cap \mathrm{int}_\gamma(\mathrm{U}) = \phi$[19].

Since X is γ-extremally disconnected, we have

$$cl_\gamma(\mathrm{int}_\gamma(\mathrm{V})) \cap \mathrm{cl}_\gamma(\mathrm{int}_\gamma(\mathrm{U})) = \phi.$$

Therefore, by Lemma 3.12, we have $x \notin cl_\gamma(\mathrm{int}_\gamma(\mathrm{U})) = \mathrm{cl}_\gamma(\mathrm{U})$.

Sufficiency. For every γ-open set U, $U \in SO_{\gamma^*}(X)$ and by Lemma 3.10 ,

$$cl_\gamma(U) = scl_{\gamma^*}(U) = \mathrm{int}_\gamma(\mathrm{cl}_\gamma(\mathrm{U})).$$

This shows that $cl_\gamma(U)$ is γ-open [18], for every $U \in \tau_\gamma$.

§4. γ-s-closed spaces

Definition 4.1. A space X is said to be γ-s-closed if for every cover $\{V_\alpha : \alpha \in I\}$ of X by γ^*-semi-open sets of X, there exists a finite subset I_0 of I such that $X = \bigcup_{\alpha \in I_0} scl_{\gamma^*}(V_\alpha)$.

Definition 4.2. A filter base Γ on X is said to γ-SR-converges to $x \in X$ if for each $V \in SR_{\gamma^*}(X)$, there exists $F \in \Gamma$ such that $F \subseteq V$.

A filter base Γ is said to be γ-SR-accumulate at $x \in X$ if $V \cap F \neq \phi$, for every $V \in SR_{\gamma^*}(X)$ and every $F \in \Gamma$.

Proposition 4.3. For any space X, the following are equivalent:

(1) X is γ-s-closed.

(2) Every cover of X by γ^*-semi-regular sets has a finite sub cover.

(3) For every family $\{V_\alpha : \alpha \in I\}$ of γ^*-semi-regular sets such that $\bigcap\{V_\alpha : \alpha \in I\} = \phi$, there exists a finite subset I_0 of I such that $\bigcap\{V_\alpha : \alpha \in I_0\} = \phi$.

(4) Every filter base γ-SR-accumulates at some point of X.

(5) Every maximal filter base γ-SR-converges to some point of X.

Proof. (1)⇒(2). Follows directly from the definition of γ-s-closed space.

(2)⇒(5). Let Γ be the maximal filter base on X. Suppose that Γ does not γ-SR-converge to any point of X. Then Γ does not γ-SR-accumulate at any point of X.

Therefore for each $x \in X$, there exist $F_x \in \Gamma$ and $V_x \in SR_{\gamma^*}(X)$ and such that $V_x \cap F_x = \phi$. The family $\{V_x : x \in X\}$ is a cover of X by γ^*-semi-regular sets of X. By (2), there exists finite number of points $x_1, x_2, x_3, \cdots, x_n$ such that $X = \bigcup\{V_{x_1} : i = 1, 2, \cdots, n\}$.

Since Γ is filter base on X, there exists $F_0 \in \Gamma$ such that $F_0 \subseteq \bigcap\{V_{x_1} : i = 1, 2, \cdots, n\}$. Therefore, we have $F_0 = \phi$.

This contradiction proves the required.

(5)⇒(4). Let Γ be a filter base on X and Γ_0 a maximal filter base such that $\Gamma \subseteq \Gamma_0$. By (5), Γ_0 γ-SR-converges to some $x \in X$. For every $F \in \Gamma$ and every $V \in SR_{\gamma^*}(X)$, there exists $F_0 \in \Gamma_0$ such that $F_0 \subseteq V$. Therefore, we obtain $V \cap F \supseteq F_0 \cap F \neq \phi$. This shows that Γ γ-SR-accumulates at x.

(4)⇒(3). Let $\{V_\alpha : \alpha \in I\}$ be a family of γ^*-semi regular sets such that $\bigcap\{V_\alpha : \alpha \in I\} = \phi$. Let $\Lambda(I)$ denot the family of all finite subsets of I. Assume that $\bigcap\{V_\alpha : \alpha \in \Omega\} \neq \phi$, for every $\Omega \in \Lambda(I)$.

Then the family $\Gamma = \left\{ \bigcap_{\alpha\in\Omega} V_\alpha : \Omega \in \Gamma(I) \right\}$ is a filter base on X. By (4), Γ γ-SR-accumulates at some $x \in X$. Since $\{X - V_\alpha : \alpha \in I\}$ is a cover of X, $x \in X - V_{\alpha_0}$ for some $\alpha_0 \in I$. Therefore, we have $X - V_{\alpha_0} \in SR_{\gamma^*}(X)$ and $V_{\alpha_0} \in \Gamma$. This is a contradiction.

(3)⇒(1). Let $\{V_\alpha \in I\}$ be a cover of X by γ^*-semi-open sets of X. By Proposition 3.7, $\{scl_{\gamma^*}(V_\alpha) : \alpha \in I\}$ is a γ^*-semi-regular cover of X.

Thus $\{X - scl_{\gamma^*}(V_\alpha) : \alpha \in I\}$ is a family of γ^*-semi-regular sets of X having the empty intersection.

By (3), there exists a finite subset I_o of I such that

$$\bigcap\{X - scl_{\gamma^*}(V_\alpha) : \alpha \in I_0\} = \phi$$

and hence $X = \bigcup\{scl_{\gamma^*}(V_\alpha \in I_0)\}$.

This shows that X is γ-s-closed. This completes the proof.

Corollary 4.4. For any space X, the following are equivalent:

(1) X is γ-s-closed.

(2) Every γ^*-semi-open and γ^*-semi-closed cover of X has a finite sub cover.

(3) Every γ^*-regular-semi-open cover of X has a finite sub cover, where γ is an open operation.

Proof. This is an immediate consequence of Proposition 3.6 and Proposition 4.3.

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A successive linear programming algorithm for SDP relaxation of binary quadratic programming[1]

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Abstract In this paper, we obtain a successive linear programming algorithm for solving the semidefinite programming (SDP) relaxation of the binary quadratic programming. Combining with a randomized method of Goemans and Williamson, it provides an efficient approximation for the binary quadratic programming. Furthermore, its convergence result is given. At last, We report some numerical examples to compare our method with the interior-point method on Maxcut problem.

Keywords Binary quadratic programming, successive quadratic programming algorithm, semidefinite programming, randomized method.

§1. Introduction

In this paper, we consider the following binary quadratic programming

$$\begin{aligned} \min \quad & x^T Q x + 2r^T x \\ \text{s.t.} \quad & x_i^2 = 1, \text{for } i = 1, \cdots, n, \end{aligned} \tag{1}$$

where Q is an $n \times n$ real symmetric matrices, r is a real n-dimensional column vectors. Without loss of generality, we assume that Q is a positive definite matrix, because of the equivalence between $\min_{x\in\{1,-1\}^n} x^T Qx + 2r^T x$, and $\min_{x\in\{1,-1\}^n} x^T Qx + 2r^T x + \sum_{i=1}^{n} y_i(x_i^2 - 1)$ for all $y \in R$. In mathematical term, it means that $Q \succ 0$.

The binary quadratic programming is a fundamental problem in optimization theory and practice. VLSI design, statistical physics, combinatorial optimization, the optimal multiuser detection and the design of FIR filters with discrete coefficients are all sources of the quadratic

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binary programming [1]−[5]. These problems are know to be NP hard [1]. One typical approach to solve these problems is to construct lower bounds for approximating the optimal value. The classic technique to obtain bounds is either via a continuous relaxation or via the dual problem, which is usually followed by branch and bound type algorithms for refining it. e.g.the paper [6] and reference therein. Recently, the semidefinite programming relaxation approach had been studied and proven to be quite powerful for finding approximate optimal solutions. See, e.g., [7] and references therein. But the semidefinite programming approach is limited to problems of moderate size, which can't solve the problems of large scale efficiently [1]. Recently, the nonlinear programming methods have been proposed to solve the large scale problem, which is more efficient. For instance, Helmberg and Rendl [7] introduce the spectral bundle method which builds on the framework of the proximal method of Kiwiel; The approach by Homer and Peinado [8] for using the change of variables $X = VV^T,\ V \subset \Re^{n\times n}$, where X is the primal matrix variable of the maxcut SDP relaxation, is to transform the maxcut SDP relaxation into a constrained nonlinear programming problem in the new variable V. More recently, Samuel Burer and Renato D.C.Monteiro [9] propose a variant of Homer and Peinado's method based on the constrained nonlinear programming reformulation of the maxcut SDP relaxation obtained by using the change of variable $X = LL^T$, where L is a lower triangular matrix.

In this paper, we consider the binary quadratic programming and its corresponding reformulation of the SDP relaxation directly. A successive linear programming algorithm for solving SDP relaxation of the binary quadratic programming is provided by using the SDP relaxation and the change of variables $X = VV^T,\ V \subset \Re^{n\times n}$. Furthermore, its convergence result is given. The step-size in our algorithm is obtained by solving n easy quadratic equations without using the linear search technique. The computational experience with our method indicates that it is substantially faster than the interior-point method.

The paper is organized as follows. In Section 2, we present the binary quadratic programming problem and its relaxations. In Section 3, the successive linear programming algorithm of the relaxation problem is obtained and its convergence result is given. Some numerical examples are offered in the last section.

Notation and Terminology.

In this paper, $\Re$, $\Re^n$,and $\Re^{n\times n}$ denote the space of real numbers, real n-dimensional column vectors, and real $n \times n$ matrices, respectively. By S^n we denote the space of real $n \times n$ symmetric matrices, and we definite S^n_+ and S^n_{++} to be the subsets of S^n consisting of the positive semidefinite and positive definite matrices respectively. We write $A \succeq 0$ and $A \succ 0$ to indicate that $A \in S^n_+$ and $A \in S^n_{++}$, respectively. We let $\mathrm{tr}(A)$ denote the trace of a matrices $A \in \Re^{n\times n}$, we defined $A \bullet B = \langle A,\ B\rangle = \mathrm{tr}(A^T B)$, and the Frobenius norm of $A \in \Re^{n\times n}$ is defined to be $\| A \|_F = (A \bullet A)^{1/2}$. We adopt the convention of denoting matrices by capital letters and matrices entries by lowercase letters with double subscripts. For example, a matrix $A \in \Re^{n\times n}$ has entries a_{ij} for $i, j = 1, \cdots, n$, In addition, we denote the rows of a matrix by lowercase letters with single subscripts. For example, $A \in \Re^{n\times n}$ has rows a_i for $i = 1, \cdots, n$. In this paper, we will often find it necessary to compute the dot product of two row vectors a_i and b_j which arise as rows of the matrices A and B. Instead of denoting this dot product as $a_i b_j^T$, we will denote it as $\langle a_i, b_j\rangle$.

§2. Binary quadratic programming and its relaxations

In this section, we consider the following binary quadratic programming and describe some of its relaxations.

$$\begin{array}{ll} \max & x^T C x \\ \text{s.t.} & x_i^2 = 1, \text{for } i = 1, \cdots, n. \end{array} \tag{1}$$

Without loss of generality, we assume that C is a positive definite matrix, because of the equivalence between $\text{Max}_{x\in\{1,-1\}^n} x^T C x$ and $\text{Max}_{x\in\{1,-1\}^n} x^T C x + \sum_{i=1}^{n} y_i(x_i^2 - 1)$ for all $y \in \Re$. In mathematical term, it means that $C \succ 0$.

Let $X = xx^T$, the above problem equals the following problem,

$$\begin{array}{ll} \max & C \bullet X \\ \text{s.t.} & (e_i e_i^T) \bullet X = 1, \text{ for } i = 1, \cdots, n \\ & \text{rank}(X) = 1 \\ & X \succeq 0, \end{array} \tag{2}$$

where e_i is the unit vector whose i-th component is 1 and others are all 0.

Because rank one constraint is nonconvex, dropping rank one constraint yields a semidefinite programming relaxation of (1) as follows,

$$\begin{array}{ll} \max & C \bullet X \\ \text{s.t.} & (e_i e_i^T) \bullet X = 1, \text{ for } i = 1, \cdots, n \\ & X \succeq 0. \end{array} \tag{3}$$

Now, we consider the following problem,

$$\begin{array}{ll} \max & C \bullet X \\ \text{s.t.} & (e_i e_i^T) \bullet X \leq 1, \text{for } i = 1, \cdots, n \\ & X \succeq 0. \end{array} \tag{4}$$

Obviously, any feasible solution to the problem (3) is a feasible solution to the problem (4), say, the problem (4) is a relaxation of the problem (3). Conversely, we assumed that X is an optimal solution to the problem (4) and let Y be a matrix which is satisfied that $y_{ii} = 1, \text{for } i = 1, \cdots, n$ and $y_{ij} = x_{ij}, i \neq j, \text{for } i, j = 1, \cdots, n$. By using the $C \succeq 0$ and the fact that $x_{ii} \leq 1$ for $i = 1, \cdots, n$, we have $C \bullet X \leq C \bullet Y$. Along with that Y is a feasible solution of the problem (3), the optimal values of the problem (3) and (4) coincide. This shows the problem (3) and (4) are equivalent.

We now present the nonlinear programming reformulation of the problem (4) which is the basis of our algorithm for finding an approximate solution of the binary quadratic programming. For every $X \in S_+^n$, there exists a matrix $V \in \Re^{n\times n}$ such that $X = VV^T$. Thus the problem can be stated as the following one,

$$\begin{array}{ll} \max & C \bullet (VV^T) \\ \text{s.t.} & (e_i e_i^T) \bullet (VV^T) \leq 1, \text{for } i = 1, \cdots, n \\ & V \in \Re^{n\times n}. \end{array} \tag{5}$$

Notice that we have replaced the requirement $X \succeq 0$ with $X = VV^T, V \in \Re^{n\times n}$. So the objective function of the problem (5) is nonconvex, but the feasible set of the problem (5) is convex.

§3. The successive linear programming algorithm to the relaxation problem

In this section, we develop and discuss the successive linear programming algorithm to solve the problem (5). Before giving the basic steps of the algorithm, we definite some functions as follows.

$$f : \Re^{n\times n} \mapsto \Re, f(V) = C \bullet (VV^T)$$

$$g_i : \Re^{n\times n} \mapsto \Re, g_i(V) = e_i e_i^T \bullet (VV^T) - 1, i = 1, \ldots, n$$

Obviously, the gradient of function $f(V)$ at a point V is $G = 2CV$, the gradient of function $g_i(V)$ at a point V is $H_i = 2(e_i e_i^T)V, i = 1, \cdots, n$.

Given a matrix V^k feasible for the problem (5), the feasible ascent direction D^k of the function $f(V)$ at a point V^k will be obtained by solving the following quadratic programming.

$$\begin{aligned} \max \quad & t \\ \text{s.t.} \quad & -G^k \bullet D + t \le 0 \\ & (e_i e_i^T) \bullet (V^k (V^k)^T) - 1 + 2(e_i e_i^T)V^k \bullet D + t \le 0, \text{for } i = 1, \cdots, n, \end{aligned} \tag{6}$$

where $(t, D) \in \Re \times \Re^{n\times n}$.

Proposition 3.1. Given $(t^k, D^k) \in \Re \times \Re^{n\times n}$ is a optimal solution for problem (6), and if $D^k \neq 0$, then D^k is the feasible ascent direction D^k of the function $f(V)$ at a point V^k.

Proof. Since $t = 0, D = 0$ is a feasible solution for problem (6), (t^k, D^k) is a optimal solution for problem (6), and $D^k \neq 0$, so we have $t^k - \frac{u}{2} D^k \bullet D^k \ge 0$, that is to say $t^k \ge \frac{u}{2} D^k \bullet D^k > 0$. Furthermore,since $-G^k \bullet D^k + t^k \le 0$, we have $G^k \bullet D^k \ge t^k > 0$. From the assumption $D^k \neq 0$, we obtain that D^k is the ascent direction of function $f(V)$ at a point V^k. Next, we will prove D^k is the feasible direction.

Let $\delta_i > 0, i = 1, \cdots, n$, and we have

$g_i(V^k + \delta_i D^k) = (e_i e_i^T) \bullet (V^k + \delta_i D^k)(V^k + \delta_i D^k)^T - 1$

$$= (e_i e_i^T) \bullet V^k (V^k)^T - 1 + 2\delta_i (e_i e_i^T)V^k \bullet D^k + \delta_i^2 (e_i e_i^T) \bullet D^k (D^k)^T.$$

If $(e_i e_i^T)V^k \bullet D^k \ge 0$, let $\delta_i < 1$, and satisfy $\delta_i^2 (e_i e_i^T) \bullet D^k (D^k)^T < t^k$, We obtain

$$g_i(V^k + \delta_i D^k) < (e_i e_i^T) \bullet (V^k (V^k)^T) - 1 + 2(e_i e_i^T)V^k \bullet D + t \le 0.$$

If $(e_i e_i^T)V^k \bullet D^k < 0$, let δ_i satisfy $2\delta_i (e_i e_i^T)V^k \bullet D^k + \delta_i^2 (e_i e_i^T) \bullet D^k (D^k)^T < 0$, we obtain

$$0 < \delta_i \le \frac{-2(e_i e_i^T)V^k \bullet D^k}{(e_i e_i^T) \bullet D^k (D^k)^T}.$$

We have $g_i(V^k + \delta_i D^k) \leq 0$. Based on the analysis, we select $\delta = \min\{\delta_i, i = 1, \cdots, n\}$, which must satisfy

$$g_i(V^k + \delta D^k) \leq 0.$$

Furthermore, since the feasible region of problem (5)is convex, so D^k is the feasible ascent direction D^k of the function $f(V)$ at a point V^k.

By proposition 3.1, we obtain the feasible ascent direction D^k of the function $f(V)$ at a point V^k. Now, we will give a simple method to obtain an appropriate step-size.

From proposition 3.1, we know that if $D^k \neq 0$, there is a step-size $\delta \geq 0$, which satisfy $g_i(V^k + \delta D^k) \leq 0$, for $i = 1, \cdots, n+1$.

Given $\delta_i \geq 0$, by solving the equations $g_i(V^k + \delta D^k) = 0$, for $i = 1, \cdots, n+1$. We obtain

$$\delta_i = \frac{-(e_i e_i^T)V^k \bullet D^k + \sqrt{((e_i e_i^T)V^k \bullet D^k)^2 - ((e_i e_i^T) \bullet V^k(V^k)^T - 1)((e_i e_i^T) \bullet D^k(D^k)^T)}}{(e_i e_i^T) \bullet D^k(D^k)^T}.$$

We choose the step-size

$$\delta = \min\{\delta_i, i = 1, \cdots, n\}. \tag{7}$$

Proposition 3.2. Given $D^k \neq 0$, the step-size from (8) is an appropriate step-size.

Proof. Given scalar $h > 0$, We define $\varphi : \Re \mapsto \Re$ by $\varphi(h) = f(V^k + hD^k)$, and we have

$$\varphi(h) = f(V^k + hD^k) = f(V^k) + h\mathrm{tr}(G^k D^k) + h^2 C \bullet D^k(D^k)^T.$$

$$\varphi'(h) = \mathrm{tr}(G^k D^k) + 2hC \bullet G^k(G^k)^T.$$

Based on the proposition 3.1 and $D^k \neq 0$, we have $\mathrm{tr}(G^k D^k) > 0$. Furthermore, since $C \succ 0$, so $C \bullet G^k(G^k)^T \geq 0$, Thus, when $h > 0, \varphi'(h) \geq 0,$ then $\varphi(h)$ is a monotone increasing function. Directly following from (7), the step-size is an appropriate search step-size.

We are now ready to write the successive linear programming algorithm.

The successive linear programming algorithm.

Let V^0 be a feasible solution of the problem (5), which is satisfied that $(e_i e_i^T) \bullet (V^0(V^0)^T) = 1,$ for $i = 1, \cdots, n$. And a prespecified constant $\varepsilon > 0$. Let $u = 0.5/n$.

1. Compute the gradient $G^k = 2CV^k$ for the function f at the point V^k.

2. Compute the feasible ascent direction D^k by solving problem (6). By the proposition 3.1, we obtain D^k.

3. Compute$\|D^k\|_F$.

4. If $\|D^k\|_F < \varepsilon$, then stop; otherwise,go to 5.

5. compute δ by formula (7). Let $V^{k+1} = V^k + \delta D^k, k = k + 1$, go to 1.

Now,we will prove the convergence of the above algorithm.

Proposition 3.3. Assume that $(t^k, D^k) \in R \times R^{m \times m}$ is the optimal solution of the problem (7), If $t^k = 0$, there are some multipliers μ_i^k, which satisfy

$$2CV^k = 2\sum_{i=1}^{n} \mu_i^k (2e_i e_i^T)V^k$$

$$\mu_i^k((e_i e_i^T) \bullet V^k(V^k)^T - 1) = 0, and\ \mu_i^k \geq 0 \text{ for } i = 1, \cdots, n.$$

That is to say the matrix V^k is the KKT point of the problem (5).

Proof. The dual programming of the linear programming (6) is [10]

$$(DLP) \quad \begin{array}{ll} \max & \sum\limits_{i=2}^{m+1} \lambda_i (1 - (e_{i-1}e_{i-1}^T) \bullet V^k (V^k)^T) \\ \text{s.t.} & \sum\limits_{i=1}^{m+1} \lambda_i = 1 \\ & \lambda_1 G^k + \sum\limits_{i=2}^{m+1} \lambda_i (2e_{i-1}e_{i-1}^T) V^k = 0 \\ & \lambda_i \geq 0, i = 1, \cdots, m+1. \end{array} \tag{8}$$

Because $t^k = 0$, due to the dual theory of the linear programming, we have that there is an solution $\lambda^k \in R^{m+1}$ for problem (8), which satisfy

$$\sum_{i=2}^{m+1} \lambda_i^k (1 - (e_{i-1}e_{i-1}^T) \bullet V^k (V^k)^T) = 0, \ \lambda_i \geq 0, \ i = 1, \cdots, m+1, \tag{9}$$

$$\lambda_1^k G^k + \sum_{i=2}^{m+1} \lambda_i^k (2e_{i-1}e_{i-1}^T) V^k = 0, \tag{10}$$

If $\lambda_1^k \neq 0$, let

$$\mu_i^k = \frac{\lambda_{i+1}^k}{\lambda_1^k}, i = 1, \cdots, m+1.$$

By the equation (10), we have

$$G^k + \sum_{i=1}^{m} \mu_i^k (2e_i e_i^T) V^k = 2CV^k + \sum_{i=1}^{m} \mu_i^k (2e_i e_i^T) V^k = 0.$$

Based on the (9), we obtained that

$$\mu_i^k (e_i e_i^T \bullet V^k (V^k)^T - 1) = 0, \mu_i^k \geq 0, i = 1, \cdots, m+1.$$

If $\lambda_1^k = 0$, following from (10), we have

$$\sum_{i=2}^{m+1} \lambda_i^k (2e_{i-1}e_{i-1}^T) V^k = 0.$$

Thus, we have

$$\sum_{i=2}^{m+1} \lambda_i^k (2e_{i-1}e_{i-1}^T) V^k \bullet V^k = 0.$$

Following from (9), we have

$$\sum_{i=2}^{m+1} \lambda_i^k (1 - e_{i-1}e_{i-1}^T \bullet V^k (V^k)^T) = \sum_{i=2}^{m+1} \lambda_i^k = \sum_{i=1}^{m+1} \lambda_i^k = 0.$$

Because λ^k is the solution of problem (8), then we have $\sum\limits_{i=1}^{m+1} \lambda_i^k = 1 \neq 0$, which contradict the equation above, so $\lambda_1^k \neq 0$. We have proven the theory.

Based on the proposition 3.3, the algorithm will terminated at the KKT point of problem (5) if the termination criterion is $t^k < \varepsilon$. Where ε is a constant which is enough small.

Not every KKT point of the problem (5) is a global solution. The following proposition give sufficient conditions for a KKT point of the problem to be a global solution.

Proposition 3.4. Assume that V^k is a KKT point of problem (5) and Let

$$\mu_i^k \geq 0, \text{ for } i = 1, \cdots, n; S^k = \sum_{i=1}^{n} \mu_i^k (e_i e_i^T) - C.$$

If $S^k \succeq 0$, then $V^k (V^k)^T$ is a global solution to the problem (4),

Proof. Since V^k is a KKT point, together with Proposition 3.4, we have

$$G^k = 2CV^k = \sum_{i=1}^{n} \mu_i^k (e_i e_i^T) V^k, \mu_i^k \geq 0, \text{ for } i = 1, \cdots, n.$$

Thus

$$\langle CV^k, V^k \rangle = \langle \sum_{i=1}^{n} \mu_i^k (e_i e_i^T) V^k, V^k \rangle = \sum_{i=1}^{n} \mu_i^k.$$

That is $\langle C,\ V^k (V^k)^T \rangle = \sum\limits_{i=1}^{n} \mu_i^k$. If $S^k \succeq 0$, let X be feasible for (4), then $\langle S^k,\ X \rangle \geq 0$, that is $\sum\limits_{i=1}^{n} \mu_i^k \geq \langle C, X \rangle$. Therefore, $V^k (V^k)^T$ is a global solution of the problem (4).

§4. Numerical results

In this section we present computational results by comparing our method with earlier method to find approximate solutions to the maxcut problem based on solving its SDP relaxation. As stated in the introduction, the purpose of the results presented here are to show that our successive linear programming algorithm is considerably faster than interior-point method.

Maxcut problem.

The maxcut problem is one of the standard $NP-$complete problems defined on graphs[11]. Let $G = (V, E)$ denote an edge-weighted undirected graph without loops or multiple edges. We use $V = \{1, \cdots n\}$, ij for an edge with endpoints i and j, and a_{ij} for the weight of an edge $ij \in E$.

For $S \subseteq V$ the cut $\delta(S)$ is the set of edges $ij \in E$ that have one endpoint in S and the other endpoint in $V \setminus S$. The maxcut problem asks for the cut maximizing the sum of the weights of its edges. Here, we only work with the complete graph K_n. In order to model a graph in this setting, define $a_{ij} = 0$ for $ij \notin E$. $A = (a_{ij}) \in S^n$ is referred to as the weighted adjacency matrix of the graph. An algebraic formulation can be obtained by introducing cut vectors $x \in \{-1, 1\}^n$ with $x_i = 1$ for $i \in S$ and $x_i = -1$ for $i \in V \setminus S$.

The maxcut problem can be formulated as the integer quadratic program.

$$\begin{aligned} \max \quad & \tfrac{1}{2} \sum_{i<j} a_{ij}(1 - x_i x_j) \\ s.t. \quad & x_i \in \{-1, 1\}, i = 1, \cdots, n. \end{aligned}$$

The matrix $L(G) = Diag(Ae) - A$ is called the Laplace matrix of the graph G. Where e is the unit vector whose every component is 1. And $Diag(Ae)$ is the diagonal matrix whose diagonal elements are Ae. Let $C = \frac{1}{4}L$, the maxcut problem may be interpreted as a special case of the problem (1).

Numerical examples.

We report the numerical example in this section. In the numerical example, we compare the computational results between our method and interior point method [5].

As stated before, the purpose of the results presented here is to show that our algorithm is substantially faster than interior point method. All the algorithms are run in the MATLAB 6.1 environment on a AMD AthlonXP1600+ personal computer with 128Mb of Ram.

In all the test problems, we choose the initial iterate L^0 to be $n \times n$ identity matrix, u to be $0.5/n$. In interior-point method, we solve the SDP relaxation by using SDPpack software [12]. We adopt the randomized cut generation scheme of Goemans and Williamson [4]. The iteration number of this algorithm is n which is the scale of max-cut problem considered. Here all the tested problems are random graphs with two different edge density 0.7 and 0.3, which denote the dense random graphs and sparse random graphs respectively.

We select problems in size from $n = 50$ to $n = 300$ to compare the suboptimal value of maxcut problem,and the total time of the three methods. In our algorithm, the iteration stops once $t^k < \varepsilon$ is found. The result is shown in Table.1.

In Table 1, we use "SDP" presents for interior point algorithm based on semidefinite programming, "SLA" for our successive linear programming algorithm, "time" for the total time of two methods, "value-f" for the suboptimal value of the maxcut problem based these methods, "density" for edge density of the random graphs. The Table 1 shows our method can generally reach solutions of the problems much faster than the interior-point method whether to the dense random graphs or the sparse random graphs.

Table.1 Comparison of the two method

Size of graph	density	algo	value-f	time
50	0.7	SLA	381.316300729441	0.1720
		SDP	381.222818884658	1.2970
50	0.3	SLA	384.343853121280	0.4220
		SDP	384.343853121280	3.2190
100	0.7	SLA	1385.94015100020	1.4530
		SDP	1388.66226246641	13.7190
100	0.3	SLA	7179.35285955398	10.0000
		SDP	7184.64381700424	149.2200
150	0.7	SLA	3027.47658904186	3.5470
		SDP	3030.60112063104	75.6090
150	0.3	SLA	1527.69134585512	3.7500
		SDP	1528.54917463768	82.5470
200	0.7	SLA	5323.36289231888	18.3750
		SDP	5318.11148873339	184.5470
200	0.3	SLA	2676.47956117645	15.1250
		SDP	2672.86837035170	200.8750
250	0.7	SLA	8156.16895709477	45.4850
		SDP	8162.65641905066	440.6560
250	0.3	SLA	4129.85944356911	14.8750
		SDP	4135.51356364934	480.8120
300	0.7	SQA	11686.0672820264	100.1720
		SDP	11697.2745314466	2116.0620
300	0.3	SLA	5884.30709295320	80.6100
		SDP	5889.15827541389	1892.8590

§5. Conclusion

In this paper, we have proposed a successive linear programming algorithm for solving SDP relaxation of the binary quadratic programming. In the algorithm, we give a simple method for selecting the step-size. Using the randomized cut procedure of Goeman and Williamson, it can give a sub-optimal of max-cut problem. It is able to obtain a moderately accurate solution more quickly than interior point method. This paper has demostrated the single case of max-cut SDP relaxation, but we believe that the same results are apt to hold elsewhere.

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A note on primes of the form a^2+1

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Abstract In this note I prove using an algebraic identity and Wilson's Theorem that if a^2+1 is an odd prime, thus this prime must has the form $4k^2+1$, then $5 \nmid 2k-3$.

Keywords Pseudo Smarandache function, mean value, asymptotic formula.

If $n=a^2+1$ is prime and $n \neq 2$, then n is odd, thus a^2 is even and n must has the form $4k^2+1$, where $k \geq 1$ is an integer. The integers $4k^2+1$ can be written as

$$4k^2+1=(2k-3)^2+3(4k-3)+1. \tag{1}$$

If $2k-3=-1$ then $k=1$, and $5 \nmid -1$. If $2k-3=1$ then $k=2$, 17 is a prime with $(2\cdot 2-3,5)=1$. If $2k-3>1$ then

$$\begin{aligned} 4k^2+1 &\equiv 0+3(2k)+1(\bmod 2k-3) \\ &\equiv 3(2k-3)+9+1(\bmod 2k-3) \\ &\equiv 10\ (\bmod 2k-3). \end{aligned}$$

By Wilson's Theorem

$$(4k^2)! \equiv -1(\bmod 4k^2+1). \tag{2}$$

Thus exists an integer c such that $(4k^2)!+1=c\cdot(4k^2+1)$, since $4k^2>2k-3$ for all k, then $2k-3|(4k^2)!$, thus

$$0+1 \equiv c\cdot 10(\bmod 2k-3). \tag{3}$$

Then there are integers s and t, such that

$$10s+(2k-3)t=1, \tag{4}$$

thus $(5,2k-3)=1$, by contradiction if $5|2k-3$, then $0+0\equiv 1(\bmod 5)$. Thus I've proved the following

Proposition. If a^2+1 is an odd prime different of 5, then $(a-3,5)=1$.

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The natural partial order on U-semiabundant semigroups[1]

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Abstract The natural partial order on an U-semiabundant semigroup is introduced in this paper and some properties of U-semiabundant semigroups are investigated by the natural partial order. In addition, we also discuss a special class of U-semiabundant semigroups in which the natural partial order is compatible with the multiplication.

Keywords U-semiabundant semigroups, natural partial orders.

§1. Introduction

In generalizing regular semigroups, a generalized Green relation $\widetilde{\mathcal{L}}^U$ was introduced by M. V. Lawson [4] on a semigroup S as follows:

Let E be the set of all idempotents of S and U be a subset of E. For any $a, b \in S$, define

$$(a,b) \in \widetilde{\mathcal{L}}^U \quad \text{if and only if} \quad (\forall\, e \in U) \quad (\, ae = a \Leftrightarrow be = b\,);$$

$$(a,b) \in \widetilde{\mathcal{R}}^U \quad \text{if and only if} \quad (\forall\, e \in U) \quad (\, ea = a \Leftrightarrow eb = b\,).$$

It is clear that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$ and $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}^U$.

It is easy to verify that if S is an abundant semigroup and $U = E(S)$ then $\mathcal{L}^* = \widetilde{\mathcal{L}}^U$, $\mathcal{R}^* = \widetilde{\mathcal{R}}^U$; if S is a regular semigroup and $U = E(S)$ then $\mathcal{L} = \widetilde{\mathcal{L}}^U$, $\mathcal{R} = \widetilde{\mathcal{R}}^U$.

Recall that a semigroup S is called U-semiabundant if each $\widetilde{\mathcal{L}}^U$-class and each $\widetilde{\mathcal{R}}^U$-class contains an element from U.

It is clear that regular semigroups and abundant semigroups are all U-semiabundant semigroups.

The natural partial order on a regular semigroup was first studied by Nambooripad [7] in 1980. Later on, M. V. Lawson [1] in 1987 first introduced the natural partial order on an abundant semigroup. The partial orders on various kinds of semigroups have been investigated by many authors, for example, H. Mitsch [5], Sussman [6], Abian [8] and Burgess[9].

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In this paper, we will introduce the natural partial order on U-semiabundant semigroups and describe the properties of such semigroups by using the natural partial order.

We first cite some basic notions which will be used in this paper. Suppose that $e,\ f$ are elements of $E(S)$. The preorders ω^r and ω^l are defined as follows:

$$e\omega^r f \Leftrightarrow fe = e \quad \text{and} \quad e\omega^l f \Leftrightarrow ef = e.$$

In addition, $\omega = \omega^r \cap \omega^l$, the usual ordering on $E(S)$.

We use $\mathcal{D}_E$ to denote the relation $(\omega^r \cup \omega^l) \cup (\omega^r \cup \omega^l)^{-1}$. Assume (S, U) is an U-semiabundant semigroup.

It will be said that U is closed under basic products if $e, f \in U$ and $(e, f) \in \mathcal{D}_E$ then $ef \in U$.

For terminologies and notations not given in this paper, the reader is referred to Howie [3].

§2. The natural partial order

Let $S(U)$ be an U-semiabundant semigroup and $a \in S$. The $\widetilde{\mathcal{L}}^U(\widetilde{\mathcal{R}}^U)$- class containing the element a will be denoted by $\widetilde{L}^U_a(\widetilde{R}^U_a)$ respectively.

We will denote an element of $\widetilde{L}^U_a \cap U$ by a^* and an element of $\widetilde{R}^U_a \cap U$ by a^+.

Recall in [4] that a right ideal I of a semigroup S is said to be an U-admissible right ideal if for every $a \in I$ we have $\widetilde{R}^U_a \subseteq I$.

For $a \in S$, we define the principal U-admissible right ideal containing a, denoted by $\widetilde{R}^U(a)$, to be the intersection of all U-admissible right ideals containing a. Similarly, we may give the definitions of an U-admissible left ideal and the principal U-admissible left ideal.

Let S be a semigroup and $x,\ y \in S$. We say that $\widetilde{R}^U_x \leqslant \widetilde{R}^U_y$ if $\widetilde{R}^U(x) \subseteq \widetilde{R}^U(y)$. A partial order on the $\widetilde{\mathcal{L}}^U$-classes can be defined in the usual left-right dual way.

Lemma 2.1. $\widetilde{R}^U_{ax} \leqslant \widetilde{R}^U_a$, for any elements a and x of S.

Proof. Clearly, the product ax lies in aS, which is the smallest right ideal containing a. Since $\widetilde{R}^U(a)$ is a right ideal containing a, we have $aS \subseteq \widetilde{R}^U(a)$. Thus $ax \in \widetilde{R}^U(a)$.

It follows immediately that

$$\widetilde{R}^U(ax) \subseteq \widetilde{R}^U(a).$$

Lemma 2.2. Let $U \subseteq E(S)$ and $e,\ f \in U$. Then $\widetilde{R}^U_e \leqslant \widetilde{R}^U_f$ if and only if $R_e \leqslant R_f$.

Proof. Suppose first that $\widetilde{R}^U_e \leqslant \widetilde{R}^U_f$. Then we immediately have $\widetilde{R}^U(e) \subseteq \widetilde{R}^U(f)$. We claim that eS is an U-admissible right ideal.

In fact, for each $a \in eS$, $a = ea$ and so, for any $b \in \widetilde{R}^U_a$, we have $b = eb \in eS$. But $\widetilde{R}^U(e)$ is a right ideal and $e \in U$, so that $eS \subseteq \widetilde{R}^U(e)$.

Since eS is an U-admissible right ideal, we have that $\widetilde{R}^U(e) = eS$. Similarly $\widetilde{R}^U(f) = fS$. It follows that $eS \subseteq fS$, that is, $R(e) \subseteq R(f)$. Hence $R_e \leqslant R_f$.

Conversely, suppose that $R_e \leqslant R_f$. Then $eS \subseteq fS$ and so $e = fx$ for some x in S^1. Thus, by Lemma 2.1, we have $\widetilde{R}^U_e = \widetilde{R}^U_{fx} \leqslant \widetilde{R}^U_f$.

Corollary 2.3. The following statements hold on an U-semiabundant semigroup $S(U)$ for any $e, f \in U$:

(i) $(e,f) \in \widetilde{\mathcal{L}}^U$ if and only if $(e,f) \in \mathcal{L}$;

(ii) $(e,f) \in \widetilde{\mathcal{R}}^U$ if and only if $(e,f) \in \mathcal{R}$.

Theorem 2.4. Let $S(U)$ be an U-semiabundant semigroup such that U is closed under basic products. Define two relations on $S(U)$ as follows:

For any x and y of $S(U)$,

$x \widetilde{\leqslant}_r y$ if and only if $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U$ and there exists an idempotent $x^+ \in \widetilde{R}_x^U \cap U$ such that $x = x^+ y$;

$x \widetilde{\leqslant}_l y$ if and only if $\widetilde{L}_x^U \leqslant \widetilde{L}_y^U$ and there exists an idempotent $x^* \in \widetilde{L}_x^U \cap U$ such that $x = yx^*$.

Then $\widetilde{\leqslant}_r$ and $\widetilde{\leqslant}_l$ are respectively two partial orders on $S(U)$ which coincide with ω on U.

Proof. We only need to prove that $\widetilde{\leqslant}_r$ is a partial order on $S(U)$ which coincides with ω on U since the proof of $\widetilde{\leqslant}_l$ is similar.

Reflexivity follows from the fact that $S(U)$ is U-semiabundant. Now suppose that $x \widetilde{\leqslant}_r y$ and $y \widetilde{\leqslant}_r x$. Then $\widetilde{R}_x^U = \widetilde{R}_y^U$ and there exist $x^+ \in \widetilde{R}_x^U \cap U$ and $y^+ \in \widetilde{R}_y^U \cap U$ such that $x = x^+ y$ and $y = y^+ x$. By Corollary 2.3, we have $x = (x^+ y^+)x = y^+ x = y$. Next, suppose that $x \widetilde{\leqslant}_r y$ and $y \widetilde{\leqslant}_r z$.

It follows that $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U \leqslant \widetilde{R}_z^U$ and there exist $x^+ \in \widetilde{R}_x^U \cap U$ and $y^+ \in \widetilde{R}_y \cap U$ such that $x = x^+ y$ and $y = y^+ z$. Thus $x = (x^+ y^+)z$ and $\widetilde{R}_{x^+}^U = \widetilde{R}_x^U \leqslant \widetilde{R}_y^U = \widetilde{R}_{y^+}^U$ which gives $R_{x^+} \leqslant R_{y^+}$ by Lemma 2.2.

It follows that $x^+ S(U) \subseteq y^+ S(U)$ and so $x^+ = y^+ x^+$. Since U is closed under basic products and $(x^+, y^+) \in \omega^r \subseteq \mathcal{D}_E$, we deduce that $x^+ y^+ \in U$. Clearly, $(x^+, x^+ y^+) \in \mathcal{R}$ and so $(x, x^+ y^+) \in \widetilde{\mathcal{R}}^U$ by Corollary 2.3. This leads to $x \widetilde{\leqslant}_r z$.

In fact, we have already shown that $\widetilde{\leqslant}_r$ is a partial order on an U-semiabundant semigroup $S(U)$. It is easy to verify that $\widetilde{\leqslant}_r$ coincides with the order ω on U.

Now the natural partial order $\widetilde{\leqslant}$ on an U-semiabundant semigroup $S(U)$ is defined by $\widetilde{\leqslant} = \widetilde{\leqslant}_r \cap \widetilde{\leqslant}_l$. We first give an alternative description of the natural partial order $\widetilde{\leqslant}$ in terms of idempotents.

Theorem 2.5. Let $S(U)$ be an U-semiabundant semigroup such that U is closed under basic products and $x, y \in S(U)$. Then $x \widetilde{\leqslant} y$ if and only if there exist idempotents e and f in U such that $x = ey = yf$.

Proof. We first prove the sufficiency part of Theorem 2.5. Suppose that $x = ey = yf$. From $x = yf$ and Lemma 2.1 we have $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U$. Choosing an idempotent x^+ in $\widetilde{R}_x^U \cap U$, we obtain that $x = x^+ x = (x^+ e)y$.

Since $ex = x$ and $(x, x^+) \in \widetilde{\mathcal{R}}^U$, we have $ex^+ = x^+$. This implies $(x^+, e) \in \omega^r \subseteq \mathcal{D}_E$. By assumption, $x^+ e \in U$. Certainly, $(x^+, x^+ e) \in \mathcal{R}$ and so $(x^+, x^+ e) \in \widetilde{\mathcal{R}}^U$ by Corollary 2.3. Thus $(x, x^+ e) \in \widetilde{\mathcal{R}}^U$. Hence $x \widetilde{\leqslant}_r y$. A similar argument shows that $x \widetilde{\leqslant}_l y$.

The necessity part of Theorem 2.5 is straightforward from Theorem 2.4.

Theorem 2.6. Let $S(U)$ be an U-semiabundant semigroup in which U is closed under basic products and $x, y \in S(U)$. Then $x \widetilde{\leqslant}_r y$ if and only if for each idempotent $y^+ \in \widetilde{R}_y^U \cap U$ there exists an idempotent $x^+ \in \widetilde{R}_x^U \cap U$ such that $x^+ \omega y^+$ and $x = x^+ y$. The dual result holds for $\widetilde{\leqslant}_l$.

Proof. Suppose that $x\widetilde{\leqslant}_r y$. Then $\widetilde{R}_x^U \leqslant \widetilde{R}_y^U$ and $x = ey$ for some idempotent $e \in \widetilde{R}_x^U \cap U$ by Theorem 2.4.

Let f be an idempotent in $\widetilde{R}_y^U \cap U$. Then $\widetilde{R}_e^U = \widetilde{R}_x^U \leqslant \widetilde{R}_y^U = \widetilde{R}_f^U$ and so, by Lemma 2.2, $R_e \leqslant R_f$. This leads to $eS(U) \subseteq fS(U)$ and so $e = fe$ giving $ef \in U$ by hypothesis. Clearly, $(e, ef) \in \mathcal{R}$ which gives $ef\widetilde{\mathcal{R}}^U e\widetilde{\mathcal{R}}^U x$ by Corollary 2.3. Hence $ef\omega f$ and $x = ey = (ef)y$, where $ef \in \widetilde{R}_x^U \cap U$.

Conversely, suppose that for each idempotent $y^+ \in \widetilde{R}_y^U \cap U$ there exists an idempotent $x^+ \in \widetilde{R}_x^U \cap U$ such that $x^+\omega y^+$ and $x = x^+y$. Then $x = y^+x^+y$ and so $\widetilde{R}_x^U = \widetilde{R}_{y^+x^+y}^U \leqslant \widetilde{R}_{y^+}^U = \widetilde{R}_y^U$. By Theorem 2.4, $x\widetilde{\leqslant}_r y$. The proof is completed.

§3. Locally V-semiadequate semigroups

In this section we want to find the conditions on an U-semiabundant semigroup $S(U)$ which make that the natural partial order $\widetilde{\leqslant}$ is compatible with multiplication of $S(U)$.

Recall in [2] that an U-semiabundant semigroup $S(U)$ is called reduced if $\omega^r = \omega^l$ on U. A reduced U-semiabundant semigroup $S(U)$ is idempotent connected(IC) if it satisfies the two equations

IC_l: For any $f \in \omega(x^*) \cap U$, $xf = (xf)^+x$;

IC_r: For any $e \in \omega(x^+) \cap U$, $ex = x(ex)^*$.

Lemma 3.1. Let $S(U)$ be a reduced U-semiabundant semigroup then

(i) If IC_l holds then $\widetilde{\leqslant}_l \subseteq \widetilde{\leqslant}_r$;

(ii) If IC_r holds then $\widetilde{\leqslant}_r \subseteq \widetilde{\leqslant}_l$.

Proof. We only need to prove (i) because the proof of (ii) is similar. If $x\widetilde{\leqslant}_l y$ then $x^*\omega y^*$ and $x = yx^*$ by the dual result of Theorem 2.6. Thus, by applying the condition IC_l, we can obtain $x = yx^* = (yx^*)^+y = x^+y$.

Certainly, $y^+(yx^*) = yx^*$ and so $y^+(yx^*)^+ = (yx^*)^+$. Since $S(U)$ is a reduced U-semiabundant semigroup, we can easily see that $x^+ = (yx^*)^+\omega y^+$. It follows from Theorem 2.6 that $x\widetilde{\leqslant}_r y$.

Lemma 3.2. Let $S(U)$ be an U-semiabundant semigroup in which U is closed under basic products and $e \in U$. Then $eS(U)e$ is a V-semiabundant semigroup, where $V = U \cap E(eS(U)e)$.

Proof. Let a be an element of $eS(U)e$ and let f be an element of U with $(f, a) \in \widetilde{\mathcal{L}}^U$. Certainly, $ae = a$ so that $fe = f$, that is, $(e, f) \in (\omega^l)^{-1} \subseteq \mathcal{D}_E$.

Since U is closed under basic products, the element $ef \in V$. Clearly, $(ef, f) \in \mathcal{L}$ so that $(ef, f) \in \widetilde{\mathcal{L}}^U$ by Corollary 2.3. It is easy to verify that $(ef, a) \in \widetilde{\mathcal{L}}^V(eS(U)e)$. This implies that each element of $eS(U)e$ is $\widetilde{\mathcal{L}}^V$-related in $eS(U)e$ to an idempotent belonging to V.

A similar result for $\widetilde{\mathcal{R}}^V$ gives us the required V-semiabundancy.

An U-semiabundant semigroup $S(U)$ is said to be $\widetilde{\mathcal{L}}^U$-unipotent if U forms a right regular band. $S(U)$ is called U-semiadequate if U forms a semilattice.

For any $e \in U$, we call $eS(U)e$ a local submonoid of $S(U)$. We shall say that $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent(locally V-semiadequate) if every local submonoid is $\widetilde{\mathcal{L}}^V$-unipotent($V$-semiadequate).

A subset A of a poset $(X, \widetilde{\leqslant})$ is said to be an order ideal if for each $a \in A$ and for any $x \in X$ with $x \widetilde{\leqslant} a$ then $x \in A$(see [1]). An U-semiabundant semigroup satisfies the congruence condition if $\widetilde{\mathcal{L}}^U$ and $\widetilde{\mathcal{R}}^U$ are right and left congruences on an U-semiabundant semigroup, respectively (see [4]).

Now we arrive at the main result of this section.

Theorem 3.3. Let $S(U)$ be an IC reduced U-semiabundant semigroup, in which U is closed under basic products, satisfying the two conditions:

(C1) For any $e \in U$, $U \cap eS(U)e$ is an order ideal of $E \cap eS(U)e$;

(C2) The congruence condition holds.

Then the natural partial order $\widetilde{\leqslant}$ is right compatible with the multiplication if and only if $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent, where $V = U \cap eS(U)e$.

Proof. Suppose first that the natural partial order $\widetilde{\leqslant}$ is right compatible and $x, y \in V$. Then $x \widetilde{\leqslant} e$ and so $xy \widetilde{\leqslant} ey = y$.

Thus, by Theorem 2.6, there exists $f \in U$ such that $xy = yf = y(yf) = yxy$. It follows that $(xy)(xy) = x(yxy) = x(xy) = xy$ and so that $xy \in E \cap eS(U)e$. According to (C1), $xy \in V$. We have shown that V forms a right regular band. But, by Lemma 3.2, the local submonoid $eS(U)e$ is V-semiabundant. Hence $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent.

Conversely, suppose that $S(U)$ is locally $\widetilde{\mathcal{L}}^V$-unipotent, that is, for any $e \in U$, $V = U \cap eS(U)e$ forms a right regular band and $a, b, c \in S(U)$ with $a \widetilde{\leqslant} b$. Then $a \widetilde{\leqslant}_r b$ and so for each idempotent$b^+ \in \widetilde{R}_b^U \cap U$ there exists an idempotent $a^+ \in \widetilde{R}_a^U \cap U$ such that $a^+ \omega b^+$ and $a = a^+ b$.

Since $(bc, (bc)^+) \in \widetilde{\mathcal{R}}^U$ and $b^+(bc) = bc$, we have $b^+(bc)^+ = (bc)^+$. By the hypothesis that U is closed under basic products, $(bc)^+ b^+ \in U \cap b^+ S(U) b^+$. Certainly, $((bc)^+ b^+, (bc)^+) \in \mathcal{R}$ so that $(bc)^+ b^+ \widetilde{\mathcal{R}}^U (bc)^+ \widetilde{\mathcal{R}}^U bc$ by Corollary 2.3.

According to (C2), $(a^+(bc)^+ b^+, ac) = (a^+(bc)^+ b^+, a^+ bc) \in \widetilde{\mathcal{R}}^U$. Since $U \cap b^+ S(U) b^+$ is a right regular band and $a^+ \omega b^+$, we have $a^+(bc)^+ b^+ \in U \cap b^+ S(U) b^+$ and $a^+(bc)^+ b^+ = (bc)^+ b^+ a^+ (bc)^+ b^+$. Again, $ac = a^+ bc = [a^+(bc)^+ b^+]bc$, where $a^+(bc)^+ b^+ \in \widetilde{R}_{ac}^U \cap U$.

Thus

$$\widetilde{R}_{ac}^U = \widetilde{R}_{[a^+(bc)^+ b^+]bc}^U \leqslant \widetilde{R}_{a^+(bc)^+ b^+}^U = \widetilde{R}_{(bc)^+ b^+ a^+ (bc)^+ b^+}^U \leqslant \widetilde{R}_{(bc)^+}^U = \widetilde{R}_{bc}^U.$$

It follows from Theorem 2.4 that $ac \widetilde{\leqslant}_r bc$.

By Lemma 3.1, we also have $\widetilde{\leqslant}_r = \widetilde{\leqslant}_l$. Hence $ac \widetilde{\leqslant} bc$, as required.

Combining Theorem 3.3 with its dual, we may obtain

Corollary 3.4. Let $S(U)$ be an IC reduced U-semiabundant semigroup in which U is closed under basic products. If

(C1) For any $e \in U$, $U \cap eS(U)e$ is an order ideal of $E \cap eS(U)e$,

(C2) $S(U)$ satisfies the congruence condition,

then the natural partial order $\widetilde{\leqslant}$ is compatible with the multiplication if and only if $S(U)$ is locally V-semiadequate, where $V = U \cap eS(U)e$.

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On the value distribution properties of the Smarandache double-factorial function

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Abstract For any positive integer n, the famous Smarandache double-factorial function $SDF(n)$ is defined as the smallest positive integer m, such that $m!!$ is divisible by n, where the double factorial $m!! = 1 \cdot 3 \cdot 5 \cdots m$, if m is odd; and $m!! = 2 \cdot 4 \cdot 6 \cdots m$, if m is even. The main purpose of this paper is using the elementary and analytic methods to study the value distribution properties of $SDF(n)$, and give an interesting mean value formula for it.

Keywords The Smarandache double-factorial function, value distribution, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer n, the famous Smarandache double-factorial function $SDF(n)$ is defined as the smallest positive integer m, such that $m!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots m, & \text{if } m \text{ is odd ;} \\ 2 \cdot 4 \cdot 6 \cdots m, & \text{if } m \text{ is even.} \end{cases}$$

For example, the first few values of $SDf(n)$ are:

$$SDF(1) = 1,\ SDF(2) = 2,\ SDF(3) = 3,\ SDF(4) = 4,\ SDF(5) = 5,\ SDF(6) = 6,$$
$$SDF(7) = 7,\ SDF(8) = 4, SDF(9) = 9,\ SDF(10) = 10,\ SDF(11) = 11,\ SDF(12) = 6,$$
$$SDF(13) = 13,\ SDF(14) = 14,\ SDF(15) = 5,\ SDF(16) = 6\ \cdots\cdots .$$

In reference [1] and [2], F.Smarancdache asked us to study the properties of $SDF(n)$. About this problem, some authors had studied it, and obtained some interesting results, see reference [3]. In an unpublished paper, Zhu Minhui proved that for any real number $x > 1$ and fixed positive integer k, we have the asymptotic formula

$$\sum_{n \le x} SDF(n) = \frac{5\pi^2}{48} \cdot \frac{x^2}{\ln x} + \sum_{i=2}^{k} \frac{a_i \cdot x^2}{\ln^i x} + O\left(\frac{x^2}{\ln^{k+1} x}\right),$$

where a_i are computable constants.

The other contents related to the Smarandache double-factorial function can also be found in references [4], [5], [6] and [7]. For example, Dr. Xu Zhefeng [4] studied the value distribution problem of the F.Smarandache function $S(n)$, and proved the following conclusion:

Let $P(n)$ denotes the largest prime factor of n, then for any real number $x > 1$, we have the asymptotic formula

$$\sum_{n\leq x}(S(n)-P(n))^2=\frac{2\zeta\left(\frac{3}{2}\right)x^{\frac{3}{2}}}{3\ln x}+O\left(\frac{x^{\frac{3}{2}}}{\ln^2 x}\right),$$

where $\zeta(s)$ denotes the Riemann zeta-function.

The main purpose of this paper is using the elementary and analytic methods to study the value distribution problem of the double-factorial function $SDF(n)$, and give an interesting asymptotic formula it. That is, we shall prove the following conclusion:

Theorem 1. For any real number $x > 1$ and any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(SDF(n)-P(n))^2=\frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right),$$

where $P(n)$ denotes the largest prime divisor of n, and all c_i are computable constants.

Now we define another function $S(n)$ as follows: Let $S(n)$ denotes the smallest positive integer m such that $n \mid m!$. That is, $S(n) = \min\{m : \ n|m!\}$. It is called the F.Smarandache function. For this function, using the method of proving Theorem 1 we can also get the following:

Theorem 2. For any real number $x > 1$ any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x}(SDF(n)-S(n))^2=\frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right).$$

§2. Proof of the theorems

In this section, we shall prove our theorems directly. First we prove Theorem 1. We separate all integers n in the interval $[1,\ x]$ into two subsets A and B as follows: $A = \{n : \ 1 \leq n \leq x,\ P(n) > \sqrt{n}\}$; $B = \{n : \ 1 \leq n \leq x,\ n \notin A\}$, where $P(n)$ denotes the largest prime divisor of n. If $n \in A$, then $n = m \cdot P(n)$ and $P(m) < P(n)$. So from the definition of A we have $SDF(2) = 2$. For any positive integer $n > 2$ and $n \in A$, $SDF(n) = P(n)$, if $2 \ \dagger \ n$.

$SDF(n) = 2P(n)$, if $2 \mid n$. From this properties we have

$$\begin{aligned}
&\sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2 \\
=\ &\sum_{\substack{2n\le x\\ 2n\in A}} (SDF(2n)-P(2n))^2 + \sum_{\substack{2n-1\le x\\ 2n-1\in A}} (SDF(2n-1)-P(2n-1))^2 \\
=\ &\sum_{\substack{n\le \frac{x}{2}\\ 2n\in A}} (SDF(2n)-P(2n))^2 = \sum_{\substack{1<n\le \frac{x}{2}\\ 2n\in A}} (2P(2n)-P(2n))^2 \\
=\ &\sum_{\substack{1<n\le \frac{x}{2}\\ 2n\in A}} P^2(2n) = \sum_{\substack{np\le \frac{x}{2}\\ p>2n}} p^2 = \sum_{n\le \frac{\sqrt{x}}{2}} \sum_{2n<p\le \frac{x}{2n}} p^2. \qquad (1)
\end{aligned}$$

By the Abel's summation formula (See Theorem 4.2 of [8]) and the Prime Theorem (See Theorem 3.2 of [9]):

$$\pi(x) = \sum_{i=1}^{k} \frac{a_i \cdot x}{\ln^i x} + O\left(\frac{x}{\ln^{k+1} x}\right),$$

where a_i $(i = 1,\ 2, \cdots, k)$ are constants and $a_1 = 1$.

We have

$$\begin{aligned}
\sum_{2n<p\le \frac{x}{2n}} p^2 &= \frac{x^2}{(2n)^2}\cdot \pi\left(\frac{x}{2n}\right) - (2n)^2\cdot \pi(2n) - 2\int_{2n}^{\frac{x}{2n}} y\cdot \pi(y)dy \\
&= \frac{x^3}{24n^3\ln x} + \sum_{i=2}^{k} \frac{b_i\cdot x^3\cdot \ln^i n}{n^3\cdot \ln^i x} + O\left(\frac{x^3}{n^3\cdot \ln^{k+1} x}\right), \qquad (2)
\end{aligned}$$

where we have used the estimate $2n \le \sqrt{x}$, and all b_i are computable constants.

Note that $\sum_{n=1}^{\infty} \frac{1}{n^3} = \zeta(3)$, from (1) and (2) we have

$$\sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2 = \frac{\zeta(3)}{24}\frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i\cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right), \qquad (3)$$

where all c_i are computable constants.

For any positive integer n with $n \in B$, it is clear that $SDF(n) \ll \sqrt{n}\cdot \ln n$ and $P(n) \ll \sqrt{n}$. So we have the estimate

$$\sum_{\substack{n\le x\\ n\in B}} (SDF(n)-P(n))^2 \ll \sum_{n\le x} n\cdot \ln^2 n \ll x^2\cdot \ln^2 x. \qquad (4)$$

Combining (3) and (4) we have

$$\begin{aligned}
\sum_{n\le x} (SDF(n)-P(n))^2 &= \sum_{\substack{n\le x\\ n\in A}} (SDF(n)-P(n))^2 + \sum_{\substack{n\le x\\ n\in B}} (SDF(n)-P(n))^2 \\
&= \frac{\zeta(3)}{24}\frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i\cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),
\end{aligned}$$

where all c_i are computable constants. This proves Theorem 1.

Now we prove Theorem 2. Note that $S(n) - P(n) = 0$, if $n \in A$; and $|S(n) - P(n)| \ll \sqrt{n}$, if $n \in B$. So from the result of the reference [4] and the proving method of Theorem 1 we have

$$\begin{aligned}\sum_{n\le x}(SDF(n)-S(n))^2 &= \sum_{n\le x}(SDF(n)-P(n))^2+\sum_{n\le x}(S(n)-P(n))^2\\ &\quad -2\sum_{n\le x}(S(n)-P(n))\cdot(SDF(n)-S(n))\\ &= \frac{\zeta(3)}{24}\frac{x^3}{\ln x}+\sum_{i=2}^{k}\frac{c_i\cdot x^3}{\ln^i x}+O\left(\frac{x^3}{\ln^{k+1}x}\right).\end{aligned}$$

This completes the proof of Theorem 2.

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On the factorial base and related counting function[1]

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Abstract In this paper, the expression of any positive integer for base of factorial numbers were studied, a new counting function related to the factorial numbers was introduced, and some exact calculating formulae for its mean value were given.

Keywords Factorial base, digital sum, calculating formula.

§1. Introduction and results

As usual, $n!$ is defined by:

$$n! = n \cdot (n-1) \cdot \cdots \cdot 2 \cdot 1.$$

It plays a very important role in the studies of theory and application of mathematics on $n!$. Therefore the various properties of $n!$ were investigated by many authors. For example, J.Stirling proved the classical Stirling formulae [1]:

$$n! = \sqrt{2\pi n}(\frac{n}{e})^n \exp(\frac{\theta_n}{12n})\,(0 < \theta_n < 1).$$

It is well-known, any integers $m(0 \leq m \leq n! - 1)$ can be uniquely written as[2]:

$$m = \sum_{i=1}^{n-1} a_i i!, \tag{1.1}$$

where $0 \leq a_i \leq i(1 \leq i \leq n-1)$.

In this paper, we are interested in the sequence $a(m)$ which counts the number of digit code in the Factorial Base. More precisely, taking $m \in \{0, 1, 2, \cdots, n! - 1\}$, we set

$$a(m) = \sum_{i=1}^{n-1} a_i,$$

and put

$$A_r(x) = \sum_{m<x} a^r(m). \tag{1.2}$$

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About (1.2), the authors disscused this problem in [3]. Now we shall use the different methods to give an exact calculating formula to (1.2) for $r = 1, 2$. That is, we prove the following Theorems:

Theorem 1. For any positive integer k, we have

$$A_1(k!) = \frac{1}{4}k(k-1)\cdot k!.$$

Theorem 2. For any positive integer k, we have

$$A_2(k!) = \frac{1}{144}k(k-1)(9k^2-5k+10)\cdot k!.$$

Theorem 3. For any positive integer N, let $N = a_1\cdot k_1! + a_2\cdot k_2! + \cdots + a_s\cdot k_s!$ with $k_1 < k_2 < \cdots < k_s, 0 < a_i \le k_i$ $(1 \le i \le s)$ under the Factorial base. Then we have

(i) $A_1(N) = a_sA_1(k_s!) + \frac{1}{2}a_s(a_s-1)\cdot k_s! + A_1(N - a_s\cdot k!) + a_s(N - a_s\cdot k_s!).$

(ii) $A_2(N) = a_sA_2(k_s!) + 2(a_s-1)A_1(k_s!) + \frac{1}{6}a_s(a_s-1)(2a_s-1)\cdot k_s! + A_2(N - a_s\cdot k!)$
$+2a_sA_1(N - a_s\cdot k!) + (N - a_s\cdot k!)a_s^2.$

§2. Proof of the theorems

In this section, we complete the proof of Theorems. First we shall prove Theorem 1.

Prove of Theorem 1. For $0 \le m < (k-1)!$, we have

$$a(m + t(k-1)!) = a(m) + t, (0 \le t \le k-1).$$

So

$$\begin{aligned} A_1(k!) &= \sum_{m<k!} a(m) \\ &= \sum_{m<(k-1)!} a(m) + \sum_{(k-1)!\le m<2(k-1)!} a(m) + \cdots + \sum_{(k-1)(k-1)!\le m<k!} a(m) \\ &= kA_1((k-1)!) + \frac{1}{2}k(k-1)\cdot(k-1)! \\ &= kA_1((k-1)!) + \frac{1}{2}(k-1)\cdot k!. \end{aligned} \tag{1}$$

(2)

Divides by $k!$ both sides in (1), we get

$$\frac{A_1(k!)}{k!} = \frac{A_1((k-1)!)}{(k-1)!} + \frac{1}{2}(k-1). \tag{3}$$

Let $S_k = \dfrac{A_1(k!)}{k!}$, from (2) we obtain

$$S_k = S_{k-1} + \frac{1}{2}(k-1). \tag{4}$$

Note that

$$A_1(1!) = \sum_{m<1!} a(m) = a(0) = 0,$$

thus $S_1 = \dfrac{A_1(1!)}{1!} = 0$, combining (3) we have

$$S_k = \frac{1}{4}k(k-1).$$

That is

$$A_1(k!) = \frac{1}{4}k(k-1)\cdot k!.$$

This proves Theorem 1.

Prove of Theorem 2. For $0 \le m < (k-1)!$, we have

$$a^2(m + t(k-1)!) = (a(m)+t)^2 = a^2(m) + 2ta(m) + t^2, (0 \le t \le k-1).$$

So from Theorem 1, we have

$$\begin{aligned}
A_2(k!) &= \sum_{m<k!} a^2(m) \\
&= \sum_{m<(k-1)!} a^2(m) + \sum_{(k-1)!\le m<2(k-1)!} a^2(m) + \cdots + \sum_{(k-1)(k-1)!\le m<k!} a^2(m) \\
&= kA_2((k-1)!) + 2A_1((k-1)!)(1+2+\cdots+(k-1)) + (k-1)!\cdot \\
&\quad (1^2+2^2+\cdots+(k-1)^2) \\
&= kA_2((k-1)!) + k(k-1)A_1((k-1)!) + \frac{1}{6}(k-1)k(2k-1)\cdot(k-1)! \\
&= kA_2((k-1)!) + k(k-1)\cdot\frac{1}{4}(k-1)(k-2)\cdot(k-1)! + \frac{1}{6}(k-1)k(2k-1)\cdot(k-1)! \\
&= kA_2((k-1)!) + \frac{1}{4}(k-1)^2(k-2)\cdot k! + \frac{1}{6}(k-1)(2k-1)\cdot k! \\
&= kA_2((k-1)!) + k!\cdot\frac{1}{12}(k-1)(3k^2-5k+4). \qquad (5)
\end{aligned}$$

Divides by $k!$ both sides in (4), we get

$$\frac{A_2(k!)}{k!} = \frac{A_2((k-1)!)}{(k-1)!} + \frac{1}{12}(k-1)(3k^2-5k+4). \qquad (6)$$

Let $T_k = \dfrac{A_2(k!)}{k!}$, from (5) we obtain

$$T_k = T_{k-1} + \frac{1}{12}(3k^3-8k^2+9k-4).$$

Since

$$A_2(1!) = \sum_{m<1!} a^2(m) = a^2(0) = 0,$$

thus $T_1 = \dfrac{A_2(1!)}{1!} = 0$, combining (6) we have

$$\begin{aligned}
T_k &= \frac{1}{12}\left\{\left[\frac{3}{4}k^2(k+1)^2-3\right] - \left[\frac{4}{3}k(k+1)(2k+1)-8\right] + \left[\frac{9}{2}(k-1)(k+2)\right] - 4(k-1)\right\} \\
&= \frac{1}{144}k(k-1)(9k^2-5k+10).
\end{aligned}$$

That is

$$A_2(k!) = \frac{1}{144}k(k-1)(9k^2-5k+10)\cdot k!.$$

This completes the proof of Theorem 2.

Prove of Theorem 3. Note that $N = a_1 \cdot k_1! + a_2 \cdot k_2! + \cdots + a_s \cdot k_s!$, and

$$A_1(t \cdot k!) = tA_1(k!) + \frac{1}{2}t(t-1)\cdot k!, (1 \le t \le k).$$

Then we have

$$\begin{aligned}
A_1(N) &= \sum_{m<N} a(m) \\
&= \sum_{m<a_s\cdot k_s!} a(m) + \sum_{a_s\cdot k_s!\le m<N} a(m) \\
&= a_s A_1(k_s!) + \frac{1}{2}a_s(a_s-1)\cdot k_s! + \sum_{0\le m<N-a_s\cdot k_s!} a(m+a_s\cdot k_s!) \\
&= a_s A_1(k_s!) + \frac{1}{2}a_s(a_s-1)\cdot k_s! + \sum_{0\le m<N-a_s\cdot k!} (a(m)+a_s) \\
&= a_s A_1(k_s!) + \frac{1}{2}a_s(a_s-1)\cdot k_s! + A_1(N-a_s\cdot k!) + a_s(N-a_s\cdot k_s!).
\end{aligned}$$

This proves (i) of Theorem 3.

(ii) For any positive integer $t(1 \le t \le k)$, we have

$$\begin{aligned}
A_2(tk!) &= \sum_{m<tk!} a^2(m) \\
&= \sum_{m<k!} a^2(m) + \sum_{k!\le m<2k!} a^2(m) + \cdots + \sum_{(t-1)k!\le m<tk!} a^2(m) \\
&= \sum_{m<k!} a^2(m) + \sum_{m<k!} a^2(m+k!) + \cdots + \sum_{m<k!} a^2(m+(t-1)k!) \\
&= \sum_{m<k!} a^2(m) + \sum_{m<k!} (a(m)+1)^2 + \cdots + \sum_{m<k!} (a(m)+(t-1))^2 \\
&= tA_2(k!) + 2(t-1)A_1(k!) + \frac{1}{6}t(t-1)(2t-1)k!.
\end{aligned}$$

So

$$\begin{aligned}
A_2(N) &= \sum_{m<N} a^2(m) \\
&= \sum_{m<a_s\cdot k_s!} a^2(m) + \sum_{a_s\cdot k_s!\le m<N} a^2(m) \\
&= A_2(a_s\cdot k_s!) + \sum_{m<N-a_s\cdot k_s!} a^2(m+a_s\cdot k_s!) \\
&= A_2(a_s\cdot k_s!) + \sum_{m<N-a_s\cdot k!} (a(m)+a_s)^2 \\
&= A_2(a_s\cdot k_s!) + \sum_{m<N-a_s\cdot k!} (a^2(m)+2a_sa(m)+a_s^2) \\
&= a_sA_2(k_s!) + 2(a_s-1)A_1(k_s!) + \frac{1}{6}a_s(a_s-1)(2a_s-1)k! + A_2(N-a_s\cdot k!) \\
&\quad +2a_sA_1(N-a_s\cdot k!) + (N-a_s\cdot k!)a_s^2.
\end{aligned}$$

This proves (ii) of Theorem 3.

Remark. For any positive integer $r \ge 3$, using our methods we can also give an exact calculating formula for $A_r(k!)$. But in these cases, the computations are more complex.

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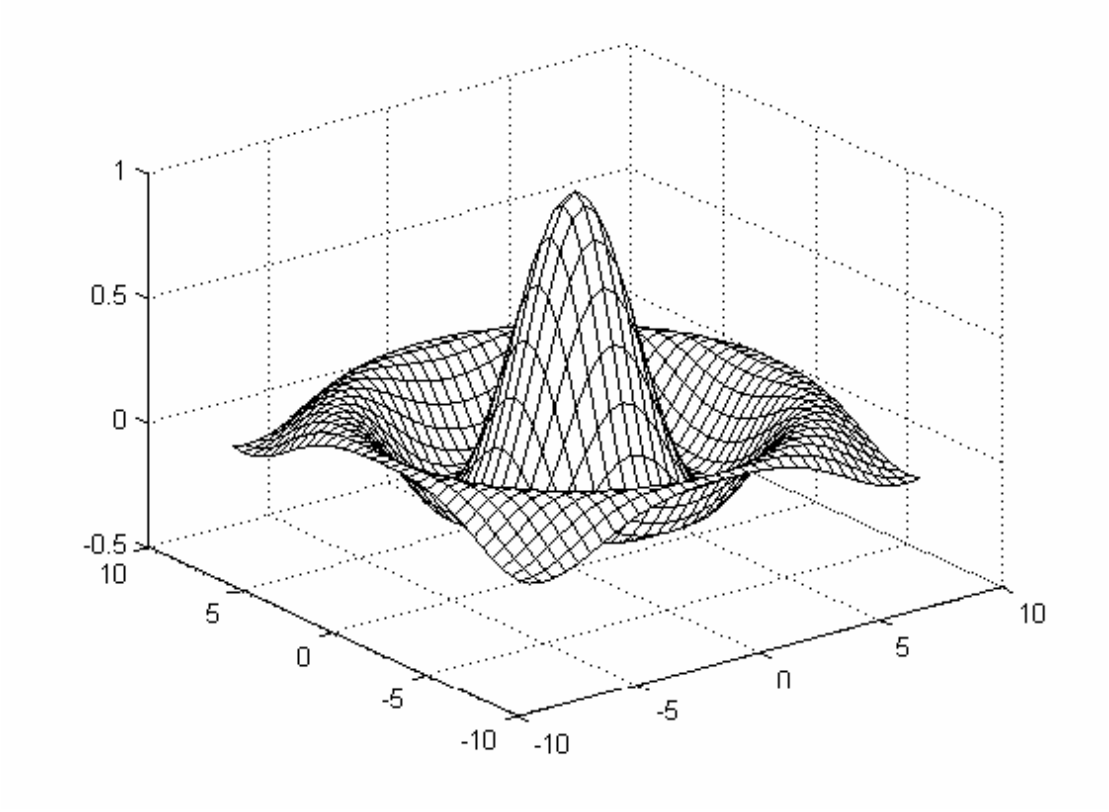

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